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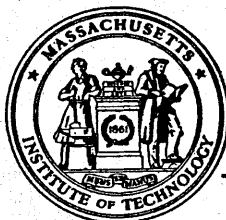
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

HYDRODYNAMICS LABORATORY

DEPARTMENT OF CIVIL AND SANITARY ENGINEERING

TECHNICAL REPORT NO. 31



THE DIFFUSION OF TWO FLUIDS OF DIFFERENT DENSITY
IN A HOMOGENEOUS TURBULENT FIELD

BY

D. R. F. HARLEMAN, J. M. JORDAAN, JR. AND J. D. LIN

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HYDRODYNAMICS LABORATORY

Department of Civil and Sanitary Engineering

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FOREWORD

The investigations described herein were conducted between February 1957 and April 1958 as part of a continuing research program dealing with the MECHANICS OF WASTE WATER DIFFUSION.

This is a final report on the first and second phases of the program:

Phase 1. Determination of the correlation between the amplitude and frequency of vibration of turbulence generating screens and the mean rate of energy dissipation by turbulence. This correlation is used to define the homogeneous turbulence level and a "mechanical" eddy viscosity.

Phase 2. To describe the diffusion characteristics of two fluids of the same or different density in a homogeneous turbulent field. Concentration measurements are obtained for the case of one-dimensional, time-dependent diffusion. The effect of a convective diffusion, due to the density difference between the two fluids, superimposed on the turbulent diffusion process is to be investigated.

The third phase, which is currently under investigation, is concerned with the distribution of diffusant concentration, in a one-dimensional turbulent river or estuary, upstream from the point of introduction of the diffusant fluid. The upstream diffusion is due to: (1) the existence of a concentration gradient in the turbulent receiving fluid and (2) the effect of a density difference between diffusant and estuary waters. The upstream diffusion is opposed by a pressure gradient due to the mean seaward flow of the estuary water. A steady-state distribution is obtained after a certain period of time measured from the beginning of diffusant discharge. The aim is to determine the effect of turbulence level, density difference and mean estuary discharge on the longitudinal concentration distribution.

The experimental investigations and analysis of data were carried out by Jan M. Jordaan, Jr., Research Assistant, who submitted his doctoral dissertation on the general material included in this report. Mr. Jordaan was assisted throughout the work by Mr. C. T. Luke, Research Assistant. Messrs. J. D. Lin and D. W. McDougall, Research Assistants, assisted in the subsequent analysis of data and in the report preparation.

The project was under the technical direction of Dr. Donald R. F. Harleman, Associate Professor of Hydraulics. Dr. Arthur T. Ippen acted as administrative supervisor for the Division of Sponsored Research of M.I.T.

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LIST OF NOTATIONS

- a = amplitude of forced vibration of turbulence generator.
- b = strand width of expanded metal screen, see sketch p. 53.
- c = concentration of diffusant, mass per unit volume.
- e = 2.72
- f = Darcy-Weisbach friction factor
- g = gravitational acceleration, 32.2 ft. /sec.²
- l = Prandtl mixing length, uncompressed length of spring.
- m = mass of diffusant contained per unit volume.
- n = number of dimensions in which diffusion takes place.
- p = resonant angular velocity, radians per second; probability.
- q = unit strength of source, mass per unit time.
- r = radius, r_0 = pipe radius.
- s = property density (general: momentum, temperature, concentration, etc.)
- t = time, second; thickness of expanded metal sheet.
- $\left. \begin{matrix} u \\ v \\ w \end{matrix} \right\}$ = local fluid velocity, $\left. \begin{matrix} u' \\ v' \\ w' \end{matrix} \right\}$ local velocity fluctuation about local mean velocity, $\left. \begin{matrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{matrix} \right\}$ local mean velocity
- $\left. \begin{matrix} x \\ y \\ z \end{matrix} \right\}$ = cartesian coordinate dimensions
- $\epsilon = \frac{\eta}{\rho}$ = kinematic eddy viscosity $\sqrt{u'^2}$, ft.²/sec.
- ζ = denotes the variable, x/\sqrt{t}
- η^2 = denotes the variable, $\frac{x^2}{2Et}$

- λ = microscale of turbulence; constant to describe concentration dependence of diffusion coefficient.
- ν = kinematic viscosity, ft.²/sec.
- $\overline{\epsilon^2}$ = variance or mean square displacement.
- π = 3.14159 ...
- ρ = mass density of fresh water at 68° F, slugs per unit volume.
- σ = frequency of forced oscillation of turbulence generator, revolutions per second.
- τ = shear stress in fluid; τ_0 = wall shear stress.
- ω = angular velocity of forced oscillation of turbulence generator, radians/sec.
- D = diffusion coefficient in classical diffusion theory; D_{mol} = molecular diffusion coefficient.
- E = turbulent diffusion coefficient or "eddy diffusivity"
- E' = diffusion coefficient due to combined effects of turbulence and gravitational convection or "gross diffusivity".
- F = force, F_I = spring and inertia force, F_D = hydrodynamic drag force.
- K = radioactivity decay constant, 1/time.
- L_{11} = mean eddy scale of turbulence, L_h = Lagrangian scale of turbulence.
- M = total mass of diffusant initially introduced; mass in spring - mass system.
- M_x } = mesh dimensions of expanded metal screen, inches.
 M_z }
- N = time rate of transfer of substance per unit area, $\frac{M}{L^2 T}$
- P = power input in ft. lbs. per sec.
- Q = discharge rate cub. ft. per sec.
- R = hydraulic radius; R = Reynolds Number; quantity under radical p. 39
- R_h = Lagrangian correlation coefficient.

S = amount of diffusant diffused into receiving fluid in time t , units of mass.

S_e = slope of energy gradient. ΔS = diffusant emitted from unit source in time Δt .

U = average velocity over cross section of conduit; U_o = center line velocity in conduit.

V = migration velocity ft./sec; V_f = convective (fresh water) velocity, ft./sec.

\bar{W} = mean rate of energy dissipation per unit volume.

I INTRODUCTION

Significance of turbulent diffusion in waste water disposal by dilution.

Industrial communities situated near large bodies of water or in drainage systems connected with such bodies dispose their waste water after treatment by dilution. Disposal by irrigation or evaporation after removing the solids by filtering, drying and incineration is justified in circumstances where the necessity outweighs the increased cost. However, comparatively few large industrial communities in the United States are situated away from either ocean, estuary, lake or river, so that the predominant form of ultimate waste water disposal is by dilution.

In waste disposal by dilution a certain degree of primary treatment is usually required to reduce the concentrations of constituents that are toxic, odoriferous or otherwise chemically or physically detrimental or objectionable to human, animal or vegetable existence.

Industrial and other wastes, varying widely in composition, coming from a diversity of establishments, such as dye or fertilizer factories, paper mills; primary treated sewage and supernatant liquor from digested sewage, radioactive waste products; wastes from hospitals, dairies, slaughter houses, etc., present different treatment problems and different standards for their effective disposal. After a sufficient time interval has elapsed following disposal, harmful chemicals will be oxidized to well below allowable levels, organic material digested by bacterial action, low level radioactive waste products; subjected to decay and a natural balance will be obtained. This can, however, be achieved only if the dilution process is aided by dispersion with currents due to winds and tidal action. Conversely, inadequate primary treatment or initial dilution can lead to widespread contamination by dispersion of harmful constituents endangering life or property. The disposal of the water-borne waste products should be made in such a fashion and at such regions in the body of water that tendencies for segregation of the influent will be minimized.

The nature of the solution of this problem is twofold: (1) the achievement of optimum mixing characteristics with economical energy input at the disposal point, (2) the location of the disposal area in a region where hydrographic or oceanographic evidence indicates degrees of boundary shear, of wave and wind generated turbulence, and thermal or tidal convection currents that will continue the dispersion of the diluted effluent in order that concentrates would not tend to accumulate with passage of time or segregate into tidal backwaters or be absorbed by vegetation or soil on the shores.

Allied problems, which have in most cases direct bearing on the flushing of disposal areas, are the salt water intrusion into river mouths and the fresh and salt water balance in tidal estuaries. Apart from the estuary flushing, there is also the consideration of contamination of public or industrial water supply intakes due to salinity intrusion.

All of the problems mentioned above, in general terms, involve the mechanics of mass transfer according to the combined operation of turbulent diffusion and convection. Turbulent diffusion processes thus fall into two general categories. In the first, the turbulent diffusion is due entirely to the momentum of the diffusant which is being introduced into a quiescent diffusing medium: this process being governed by the mechanics of momentum and mass transfer in submerged turbulent jets. In the second category, the turbulent diffusion is due largely to the turbulent energy of the receiving fluid, the diffusant being introduced without materially increasing the turbulent activity at the region of introduction. In practice the ideal dilution process would be a combination of the two processes in the above sequence. The diffusing substance would be discharged with as high a momentum as practical into the receiving medium in the form of submerged jets, and the diffusion process in the vicinity of the disposal points would be entirely governed by the energy of introduction of the diffusant. At sufficiently large distances from its source the momentum of a jet would have decayed to levels comparable to the turbulence level in the receiving body of fluid. Further dispersion will occur according to mechanics of diffusion due to the turbulence in the receiving fluid body itself. (That is, if one considers momentarily turbulence as including all sizes of eddies present and hence also what would be customarily considered convection currents). The analysis will be simplified, however, if it is considered that the motion of the fluid body consists of a field of homogeneous turbulence in which a convection pattern may be superimposed. With further simplification, the general problem may be made feasible for mathematical and experimental analysis in particular cases. Thus all of the above-enumerated disposal problems involve ultimately the mechanism of turbulent (eddy) diffusion which can accordingly be treated in two distinct parts.

1. Diffusion by submerged jets.

Since dilution is here effected as a result of turbulence produced by momentum-transfer to and entrainment of the receiving fluid, this is essentially a localized process which will cease at such a distance from the source where complete viscous dissipation of the energy of the smallest eddies occur. The ultimate dilution versus distance relationship is a function of the initial jet diameter and velocity. Many small jets will diffuse more rapidly with distance than fewer large jets but the latter would carry the diffusant further into the body of fluid where the plume may be better dispersed by currents. Provided the body of fluid is of sufficient extent in breadth as well as depth so as to prevent finite concentration reaching the boundaries, a few large jets are better than either a single jet or a multiplicity of small ones for efficient ultimate disposal. Jet diffusion has been extensively studied by others so that this investigation will be largely concerned with the second category.

2. Diffusion in a homogenous field of turbulence.

In this process diffusion occurs over large distances since the eddy scales involved vary from very small to very large. Where density differences are not large enough to cause stratification, the large scale eddies will stir regions

of higher waste-concentrations into the receiving fluid and small scale turbulence and molecular diffusion will eventually iron out all concentration gradients. The existence of a field of turbulence in the receiving body may be due to one or more of the following effects: turbulence arises in rivers due to boundary-shear on the flow resulting in a velocity gradient, and hence in local circulations, cross-stream momentum transfer and turbulence generation. In river mouths and tidal estuaries, aside from the boundary shear generated turbulence, due to the tidal oscillations, the tidal inflow causes turbulent momentum transfer with the outflowing fresh water current. Turbulence in the ocean arises from wave and tidal action on coasts, shoals, bars, and reefs; from wind generated shear at the surface, and temperature or density currents and counter-currents. It is found from a survey of available literature on turbulent diffusion in natural bodies of water, e.g. ocean, lake or estuary that the diffusion process has usually been assumed analogous to molecular diffusion except that the diffusion coefficients are of a much larger order of magnitude. One of the primary objectives of the current investigation was to determine to what extent this analogy is valid. This purpose was served by an experimental set-up where uniform turbulence in a fluid body could be created and the diffusion coefficient determined by means of concentration changes for certain established initial and boundary conditions.

Objectives

The broad purpose of this investigation has been the determination of the principles of the mechanics of turbulent diffusion; the establishment of the degree of validity of the analogy between uniform turbulent diffusion and molecular diffusion; the application of classical diffusion theory to turbulent material transfer processes including effects of gravitational convection due to density gradients.

The specific objectives that were pursued in the experimental program which was restricted to diffusion and convection processes associated with a uniform turbulence level, homogeneous throughout the fluid and independent of time but not necessarily isotropic in the x , y , z directions, are divided into three phases:

1. The determination of the rate of work input of the turbulence generating device (i.e. energy dissipation rate) as a function of the eddy viscosity, or rather, of a mechanical parameter to which the eddy viscosity is proportional; so as to define the characteristics of the turbulence level produced for any combination of the amplitude and frequency of oscillation of the turbulence generating element.

2. The determination of the relationship between turbulent diffusion coefficient and eddy viscosity parameter for uniform density throughout the fluid. Furthermore, the determination of gravity effects on the diffusion process due to density differences between the diffusant and the receiving medium. This phase will be concerned with time-dependent, one-dimensional diffusion with the semi-infinite boundary condition.

3. An extension of the investigation outlined in phase 2 for the steady-state diffusion process where the turbulent diffusion is opposed by a steady, uniform convective current in the receiving fluid.

Considerations as to the practical applicability of experimental investigations, influencing the choice of experimental program.

The experimental program was to be designed so as to serve as a research tool for studying basic fluid mechanics, yet on the other hand it had to be of practical significance.

For the laboratory experimental arrangement the one-dimensional turbulent diffusion into a semi-infinite medium due to a constant applied concentration potential was chosen. This was done for the reason that this boundary condition is obtainable by symmetry at the center of an unsteady, homogeneous desorption - sorption process where initially the concentration at $-\infty < x < 0$ is $2c_0$ and the concentration at $0 < x < \infty$ is zero. This is a convenient boundary condition for the determination of diffusion coefficients and is moreover suited to the available flume (See Fig. 11).

The laboratory experimental equipment is in addition well suited to represent a model of a particular steady-state disposal and diffusion problem about to be described, lending further justification to the choice of the semi-infinite one-dimension boundary condition.

Consider the diffusion of industrial waste water discharged at a certain point along a long, narrow estuary. It will be assumed that the waste water is discharged laterally from a set of nozzles uniformly distributed over the cross section of the estuary and that the estuary is of the "well-mixed" type, i.e. that there is sufficient turbulence to prevent any local stratification at the disposal point, or tendencies for bottom and top currents to form. Moreover, it will be assumed initially that the mean velocity of flow in the estuary is zero. The tidal oscillations thus cause a back and forth translation of the body of water in the estuary, the shear with the bottom, sides and stationary objects resulting in turbulence generation, which will be assumed to be of uniform intensity over the cross section and along the length of interest of the estuary. The diffusion process averaged over a tidal cycle may be therefore considered as one-dimensional and analogous to that generated by the oscillating screens in the laboratory flume. These experiments performed in the laboratory under Phase 2 yield data on the concentration distribution with time from which may be found the interrelationship of eddy viscosity (a measure of the turbulence) and the diffusion coefficient. Knowledge of these relationships would in practice enable programming the disposal of wastes from intermediate receivers so as to occur at portions of the tidal cycle when the turbulence created by tidal action would be at suitable levels and would enable adjusting disposal rates and concentrations accordingly.

Hitherto in this analogy the net flow over a tidal cycle was assumed to be zero. If this is not the case and there is a net flow seaward in the estuary of

average velocity V , which is assumed to be considerably less than the velocities due to tidal oscillations, the distribution of the concentration after a time will become more and more asymmetrical upstream and downstream. Eventually a quasi-steady state (neglecting the to and fro translation with the tide) will be reached in the concentration distribution upstream, while the concentration distribution downstream will continue to grow and asymptotically approach a uniform distribution with distance, accumulating the steady contribution of disposed waste water.

The third phase in the laboratory test program represented this condition of turbulent diffusion into a steady convection current. The experimental and analytical work on this phase is currently underway, the results will be presented in a separate report.

II SURVEY OF LITERATURE ON TURBULENT DIFFUSION

In order to provide a frame of reference and to present a systematic review of the wide field of published material embracing the subject of turbulent diffusion, the following subdivisions will be made in this survey:

- A. Diffusion with diffusant properties conservative, without convection.
 - a) General diffusion (thermal, molecular and Brownian)
Mathematics of diffusion
 - b) Turbulent diffusion
 - (i) Homogeneous, isotropic turbulence with eddy diffusivity independent of time, distance or concentration of diffusant.
 - (ii) Concentration-dependent diffusion coefficient.
 - (iii) Distance-dependent diffusion coefficient.
 - (iv) Space, time dependence of diffusion coefficient.
- B. Diffusion with convection, with conservation of chemical and physical properties.
 - a) General diffusion
 - b) Turbulent diffusion
 - (i) Free turbulence. Turbulence largely created by sources other than boundary shear of the flow.
 - (ii) Turbulent diffusion or dispersion largely created by boundary drag or shear flow.
 - (iii) Turbulent diffusion in free submerged jets.
- C. Diffusion with or without convection; action of body forces; non-conservative chemical and physical properties.
Mathematical solution for special cases

- D. (i) Mechanics of waste water disposal by dilution and diffusion.
(ii) Determination of diffusion coefficient from concentration fields or vice versa.

A. Diffusion with diffusant properties conservative, without convection

a) General diffusion (thermal, molecular and Brownian)

Under this sub-heading the transfer of the diffusing property is understood to occur from particle to particle of the diffusing medium over very small distances. In addition, the root-mean-square deviation of a typical particle of the diffusing medium about its mean position is assumed to be very small compared to the long time extent of the scale of the diffusion process.

Mathematics of diffusion.

The mathematical equation of heat conduction in a solid was formulated by Fourier (A-1), 1822:

$$q = \frac{\alpha^2}{c_v} \frac{\partial \theta}{\partial x} \quad [1]^*$$

It was first recognized by Fick (A-2), 1855, that molecular diffusion in liquids was mathematically related to heat conduction in solids and he formulated what is known as Fick's First Law:

$$N = - D \frac{\partial s}{\partial x} \quad [1a]$$

which states that the time rate of transfer of a fluid property N through unit area of a region in a fluid is proportional to the local normal gradient of concentration of the fluid property across the surface of the region. The constant of proportionality D is known as the molecular diffusion coefficient.

When the equation of continuity for incompressible fluid is applied to a small element of volume, one obtains the expression known as Fick's Second Law which has the form:

$$\frac{\partial s}{\partial t} = D \frac{\partial^2 s}{\partial x^2} \quad [2]$$

for the one-dimensional diffusion case.

In vector notation for the most general case where D can be a vector-function, Eqns. [1] and [2] become:

* See list of symbols

$$N = -D \cdot \text{grad } s \quad [3]$$

$$\frac{\partial s}{\partial t} = \text{div } (D \cdot \text{grad } s) \quad [4]$$

Particular solutions of Eqns. [3] and [4] for a wide range of boundary conditions may be found in standard references on heat conduction (Carslaw and Jaeger, A-3) and molecular diffusion in solids, liquids and gases (Barrer, A-4; Jost, A-5; Crank, A-11). The fundamental Fickian Diffusion Laws assume the following forms in coordinate notation (Jost, Chapter I) (See Table I.)

The solutions of some of these forms of the differential equation [4] for specified boundary conditions may be found in texts on partial differential equations, vide Hildebrand, A-8 (table on p. 481).

There are three simple boundary and initial conditions to be considered, as follows:

(i) Source functions of heat (or mass, or momentum) transfer.

Solutions for one, two or n dimensions of the diffusion process in which a finite quantity of heat or dosage of tracer is applied at a source and allowed to distribute itself are [Jahnke-Emde, A-6, (Appendix p. 47) and Webster, A-7, (p. 168)] of the form:

$$s = \frac{e^{-\frac{x^2}{4Dt}}}{(\pi y)^{\frac{n}{2}}}$$

where

n = number of dimensions in which diffusion occurs

s = property concentration per unit volume

y = 4Dt

x = distance from source

(ii) "Constant applied potential" function of heat, etc. transfer

These represent cases where initially the concentration of diffusing property is zero everywhere in the diffusion medium and thereafter the concentration at some boundary to the region of diffusing is raised to and maintained at a constant value. Constant potential functions are obtained by space-integration of the source functions. The form of the general solution for n dimensions is given by Webster, A-7. Exact solutions for the one-dimensional case involve the error-integral and its related functions widely used in diffusion studies (tabulated in Jahnke-Emde, A-6. p. 23).

TABLE I. FORMS OF THE CLASSICAL DIFFUSION EQUATION
IN VARIOUS COORDINATE SYSTEMS.

Law	Number of Dimensions	Coordinate System	Coordinates Notation	Vector Notation
Fick's First Law of Diffusion	One		$N = - D \frac{\partial s}{\partial x}$	$N = - \underline{D} \text{ grad } S$
Diffusion coefficient not constant				
Ficks 2nd Law of Diffusion	One		$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} (D \frac{\partial c}{\partial x})$	$\frac{\partial c}{\partial t} = \text{div} (\underline{D} \text{ grad } c)$
	Two	Axial Symmetry (Cylindrical Coordinates)	$\frac{\partial c}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} [rD \frac{\partial c}{\partial r}]$	
	Three	Spherical Symmetry (Spherical Coordinates)	$\frac{\partial c}{\partial t} = (\frac{1}{\rho^2}) \frac{\partial}{\partial \rho} [\rho^2 D \frac{\partial c}{\partial \rho}]$	
Constant diffusion coefficient				
	One		$\frac{\partial c}{\partial t} = D \nabla^2 c$	$\frac{\partial c}{\partial t} = D \cdot \text{div grad } c$
	Two	Axial Symmetry	$\frac{\partial c}{\partial t} = D [\frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r}]$	
	Three	Spherical Symmetry	$\frac{\partial c}{\partial t} = D [\frac{\partial^2 c}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial c}{\partial \rho}]$	

$$\frac{1}{p} \int_0^x e^{-t^p} dt, E_2(x) = \text{erf}(x), \text{ the Error Integral}$$

and is:

$$\frac{s}{s_0} = 1 - \text{erf} \frac{x}{\sqrt{4Dt}} \quad [6]$$

(iii) "Constant flux" functions of heat, etc. transfer.

Constant flux functions are obtained when the source functions are integrated with time, the diffusion process represented being the result of continuous application of finite quantities of diffusant at the source. The exact solutions for the two-dimensional case, related to the "Well-function" of ground water flow (Wisler and Brater, A-9, p. 233) involves the exponential-integral:

$$Ei(-x) = - \int_x^\infty \frac{e^{-\xi}}{\xi} d\xi$$

which is tabulated in Refs. A-9 and A-6. A series expansion (A-9, p. 234) or logarithmic asymptote approximation (Rouse, A-10, p. 367) may be used to simplify calculations, thus making it of great value in two-dimensional diffusion problems.

The methods of determination of diffusion coefficients will be discussed in some detail in Chapter III, Theoretical Considerations.

b) Turbulent diffusion.

(i) Homogeneous, isotropic turbulence with eddy-diffusivity independent of time, distance or concentration of diffusant.

A study of the process of mass transfer due to the transporting power of turbulent conditions in the diffusing medium, may be approached from the point of view of considering the process as a scaled-up version of molecular diffusion. The turbulent diffusion coefficient or eddy diffusivity E may be of the order of 10^6 or 10^9 times as large as the molecular diffusion coefficient D . The molecular mean free path is replaced by a length of the order of the average eddy size.

Where devices, such as stirring paddles or screens, are used for creating the turbulence, the concept of "mixing length" developed by Prandtl has practical significance. The eddy diffusivity E could be expressed as a length, representative of the geometry, times a velocity, either that of the paddle or otherwise the root-mean-square velocity of the turbulence. The eddy diffusivity appears to be a function of the turbulence intensity only and independent (in homo-

geneous turbulence) of the time and space history of the diffusion process or of the concentration of diffusant. It has been found, however, by the pioneers in modern turbulent diffusion theory (Taylor, C-1; Kampe de Fariet, C-2; Frenkiel, C-3) that these assumptions are not strictly true where the distances from the source of diffusion are of the same order as the eddy scale of the turbulence.

The second and more correct viewpoint from which turbulent diffusion processes may be studied, is founded on the statistical description of turbulence, the theory of which expresses the eddy diffusivity as a function of the correlation parameters as well as of the time and space history of the diffusion (C-3). This more exact theory has to be employed in, for example, heat and mass transfer processes between phases (i.e. across solid-liquid, solid-gas or liquid-gas interfaces) and in momentum, heat and mass transfer in boundary layers in fluids. In these cases rapid changes of the turbulent diffusion properties of the flow may occur with diffusion distance (e.g. D-15). The eddy diffusivity becomes independent of the time-space parameters and the diffusion process becomes analogous to heat diffusion in solids, i.e. molecular diffusion, only where the diffusion distances in homogeneous turbulence are large compared to the eddy scale of turbulence. It is further assumed that convection effects occur and that the eddy diffusivity is independent of concentration.

(ii) Concentration-dependent diffusion coefficient.

Crank (A-11, p. 166) gives examples of absorption and desorption curves (one dimensional semi-infinite medium, constant potential function) for cases where the diffusion coefficient is dependent on the concentration in certain ways. This is based on the formal solutions obtained by Fujita (A-11, A-13) where the diffusion coefficient is a function of concentration only. The Boltzmann (A-5, p. 37; A-12) transformation yields a new variable which may be solved for certain functional relationships of E with concentration, e.g.:

$$E = E_0 \left(1 + \frac{\lambda c}{c_0} \right)$$

Fujita's method gives an exact solution while Crank (p. 184) gives a very close approximate solution by the method of moments (See plot B in Appendix).

(iii) Distance-dependent diffusion coefficient.

Hanratty (D-11) and Sleicher (D-15) report cases where there is dependence of eddy diffusivity on the diffusion distance. For non-homogeneous non-isotropic turbulence, such as may occur, for example, in the immediate region behind grids or constrictions or other obstructions to the flow, a distance dependence of the diffusion coefficient is to be expected. In such a case a transformation of the distance scale element according to the ratio of a local to a reference eddy diffusivity may be made:

$$\Delta x' = \sqrt{\frac{E_x}{E}} \Delta x \quad [7]$$

Solutions of E from concentration distributions may be obtained by trial and error matchings with type curves drawn for appropriate functions of E with x .

(iv) Space, time-dependence of diffusion coefficient.

Although the molecular diffusion coefficient is negligible in turbulent diffusion compared to the eddy diffusivity, molecular diffusion plays an important role in that it is responsible for the ultimate evening out of concentration gradients across the small scale eddies. For laminar flow in a tube it was demonstrated by the experiments of Taylor (F-1) that the transverse (radial) molecular diffusion across the velocity gradient affects the longitudinal (axial) diffusion of the tracer substance released in the flow. The longitudinal diffusion coefficient is found to be inversely proportional to the molecular diffusion coefficient. Stirba and Hurt (D-18) pointed out that there is a complicated inter-relationship between molecular and eddy diffusion, the diffusion depending on such factors as miscibility, breakdown of diffusant and mobility.

In turbulent diffusion statistical parameters have to be used to express the turbulence intensity. The turbulent mass diffusion coefficient E_{mass} and turbulent heat diffusion coefficient E_{heat} (eddy diffusivities) are related to the eddy viscosity or momentum diffusion coefficient $\epsilon = \sqrt{u'^2} l$ by proportionality constants known as the turbulent Schmidt and turbulent Prandtl numbers. For liquids the value of $S_t = \frac{\epsilon}{E_{\text{mass}}}$, $P_t = \frac{\epsilon}{E_{\text{heat}}}$ is equal to approximately 0.7 to 0.85, Treybal (D-1). A review of turbulent mass and heat transfer processes by Opfel and Sage (C-7) sets forth the definition and use of effective eddy diffusivities. The correlation parameters in the statistical theory of turbulence have been applied to turbulent diffusion by researchers such as Dryden (C-9) who studied diffusion of heat from a line source in turbulent air and expressed $E_{\text{heat}} = l_1 \sqrt{u'^2}$, where $l_1 = \int_0^{\infty} R_t dt$, where R_t is the auto-correlation parameter. Dryden bases his expression on Taylor vorticity transport theory (C-1, C-1a).

Mickelson (D-12) compares the Lagrangian versus Eulerian correlations and Favre (D-17) presents two-point, two-time correlations.

An extension of the Taylor theory of diffusion by continuous movements is presented by Frenkiel (C-3). The differences between the statistical turbulent diffusion theory and the classical (assuming constant diffusion coefficient) is outlined and illustrated for various Lagrangian time correlation functions.

The conclusions drawn from a survey of these refined diffusion theories were that the differences from the classical diffusion theory are of importance only in cases where one is concerned with the initial phases of the turbulent diffusion process, i.e. where representative eddy scales are comparable to the diffusion distance. When the parameter $\frac{x}{\sqrt{t}}$ becomes small for $x \gg l$, the "average eddy size", the assumption of constant diffusion coefficient becomes valid for these theories.

B. Diffusion with convection, with conservation of chemical and physical properties.

a) General diffusion.

The differential equation for diffusion with convection is given by:

$$\frac{\partial c}{\partial t} + \text{grad}(\underline{u} \cdot c) = \text{div}(\underline{D} \cdot \text{grad} c) \quad [8]$$

for an incompressible fluid $\text{grad} \underline{u} = 0$, hence in coordinate notation:

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + w \frac{\partial c}{\partial z} = \frac{\partial}{\partial x} (D_x \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y} (D_y \frac{\partial c}{\partial y}) + \frac{\partial}{\partial z} (D_z \frac{\partial c}{\partial z}) \quad [8a]$$

For the case where \underline{D} is independent of time, homogeneous and non-isotropic, $D_x = k_1$, $D_y = k_2$, $D_z = k_3$, where $k_1 = k_2 = k_3$, Eqn. [8] becomes:

$$\frac{\partial c}{\partial t} + \underline{u} \cdot \text{grad} c = \underline{D} \cdot \text{div}(\text{grad} c) = \underline{D} \cdot \nabla^2 c \quad [9]$$

$$= D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} + D_z \frac{\partial^2 c}{\partial z^2} \quad [9a]$$

For $D_x = D_y = D_z = D$, Eqn. [2] becomes:

$$\frac{\partial c}{\partial t} + \underline{u} \cdot \text{grad} c = D \text{div}(\text{grad} c) = D \nabla^2 c \quad [10]$$

$$= D \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right) \quad [10a]$$

The diffusion-with-convection equations for cases where the analogy between molecular and turbulent diffusion is valid, will be considered first. The fundamental partial differential equation for diffusion-with-convection in a three-dimensional flow field of homogeneous turbulence, where the eddy diffusivity is assumed to be independent of location, concentration of diffusant or diffusion process history (vector $\underline{E} = \text{const.}$), is given by [9a] and [10a]. Writing \underline{E} for \underline{D} and temporal mean values \bar{c} for c and \bar{u} for u , etc., these become:

For non-isotropic turbulence -

$$\frac{\partial \bar{c}}{\partial t} + \bar{u} \frac{\partial \bar{c}}{\partial x} + \bar{v} \frac{\partial \bar{c}}{\partial y} + \bar{w} \frac{\partial \bar{c}}{\partial z} = E_x \frac{\partial^2 \bar{c}}{\partial x^2} + E_y \frac{\partial^2 \bar{c}}{\partial y^2} + E_z \frac{\partial^2 \bar{c}}{\partial z^2} = \underline{E} \cdot \nabla^2 \bar{c} \quad [11]$$

For isotropic turbulence -

$$\frac{\partial \bar{c}}{\partial t} + \bar{u} \cdot \text{grad} \bar{c} = E \nabla^2 \bar{c} \quad [12]$$

Equation [12] resembles the Navier-Stokes equation for incompressible viscous flow with zero pressure gradient

$$\frac{D\bar{u}}{Dt} = \frac{\partial \bar{u}}{\partial t} + u \frac{\partial \bar{u}}{\partial x} + v \frac{\partial \bar{u}}{\partial y} + w \frac{\partial \bar{u}}{\partial z} = \nu \nabla^2 \bar{u} \quad [13]$$

Whereas the Navier-Stokes equation is non-linear because of terms such as $u \frac{\partial u}{\partial x}$, the diffusion-convection equations [11] and [12] are linear since \bar{u} , \bar{v} , \bar{w} are independent of \bar{c} , hence particular solutions may be added.

For non-isotropic turbulence, with $u = f(x,t)$, $v = w = 0$, Equation [11] becomes:

$$\frac{\partial \bar{c}}{\partial t} + u \frac{\partial \bar{c}}{\partial x} = E \cdot \nabla^2 \bar{c} \quad [14]$$

Type solutions of Equation [14], which represents turbulent diffusion in an ocean current are treated by Proudman (B-3). Extensive studies of turbulent diffusion in the oceans, and experimental determination of diffusion coefficients in oceans from salinity traverses, have been made by Jeffreys (B-1) and Proudman (B-2).

For diffusion in one dimension only, with a current $u = f(x,t)$, $\frac{\partial \bar{c}}{\partial y} = 0$, $\frac{\partial \bar{c}}{\partial z} = 0$ and from Equation [14]:

$$\frac{\partial \bar{c}}{\partial t} + u \frac{\partial \bar{c}}{\partial x} = E \frac{\partial^2 \bar{c}}{\partial x^2} \quad [15]$$

At large values of t , a steady-state condition ($\frac{\partial \bar{c}}{\partial t} = 0$) will be approached where the convective transport becomes equal and opposite to the diffusive transport everywhere and Equation [15] becomes:

$$uc = E \frac{\partial \bar{c}}{\partial x} \quad [16]$$

b) Turbulent diffusion equations.

Equations [11] to [16] are based on the assumption of an eddy diffusivity E , independent of the history of the diffusion process. The implications of the statistical theory of diffusion by continuous movements (Taylor e.a.) have been noted in Section A. Further statistical formulations obtained by various methods of approach, e.g. Lagrangian, Eulerian, Random Walk are found in the literature Goldstein (C-4), Pai (C-5), Mickelson (C-12).

The gist of Goldstein's turbulent diffusion theory is the recognition of the basic difference between the molecular and turbulent diffusion process. In molecular diffusion the finite speed of property transfer (molecular velocity) imposes no upper limit to the speed with which a certain concentration may advance. In turbulent diffusion, the transfer rates are much higher due to migration of unit volumes of fluid (eddies) over mixing lengths much larger than molecular mean free paths but with a relatively slow speed of migration (less than the turbulence intensity), so that in the initial stages the concentration advance is limited by the "migration-velocity". This terminology is employed in a statistical sense for describing the random distribution of eddy migration-velocities. This simplification

led to Goldstein's "random walk" type concept of the diffusion process which assumes that each diffusing particle moves at constant speed in discontinuous steps of constant length. For a case of isotropy (no directional bias) and for no correlation between successive movements of any particle, the difference equations obtained leads to the "telegraph equation", another form of partial differential equation (hyperbolic) characteristically different from the classical diffusion equation (parabolic).

Solutions of the telegraph equation of communications engineering are known for the boundary conditions of diffusion into a semi-infinite medium due to a constant applied potential (Webster, A-7; Carson, C-8). For small diffusion distances the solution differs from the error-function

$$\frac{c}{c_0} = 1 - \operatorname{erf} \frac{x}{2\sqrt{Et}} \quad [6]$$

in that initially there is a delay time $\frac{x}{V}$ during which there occurs no concentration rise whatsoever at a station x away from the source. At this instant,

$t = \frac{x}{V}$, the concentration instantaneously increases to a finite value and then continues to increase, approaching the error-function solution for constant E at large values of t . When the distance x becomes larger the step rise at $t = \frac{x}{V}$ becomes insignificant, hence the Goldstein diffusion equation differentiates turbulent from classical diffusion theory only at small values of x and t . The more x approaches the order of the largest eddy scale, L_1 , the more marked the difference. ($\frac{Vx}{2E} = \frac{Vx}{2e/s_t} = \frac{VxSt}{2uL_1} \rightarrow \text{small}$), vide Fig. B

Turbulence in the ocean is characterized by eddies of a wide range in scale so that this initial deviation may be significant even where diffusion distances are large.

(i) Free turbulence. Turbulence largely created by sources other than boundary shear of the flow.

The Taylor-de Feriet formulation has received experimental support from investigations on mass transfer rates behind grids in wind tunnels (D-3, D-4, D-5) for cases in which density gradient effects are not encountered.

A number of experiments on turbulent diffusion in fluidized particulate beds (D-9, D-10, D-11, D-13) have been described in the Chemical Engineering literature. In this type of experiment the mass transfer rates are studied by sampling the spreading of dye or saline tracer introduced into the turbulent fluid body. Turbulence is created by the suspension of dispersed small spheres in a uniformly rising current. A list of references of experiments of this nature is given in Ref. D-19.

Hanratty (D-11) cites experiments in turbulent flow fields which

prove the history dependence over small distances of the eddy diffusivity in both mass and heat transfer as predicted by Taylor's theory. This dependence may be clarified in the light of the concept that the averaged process of turbulent diffusion over finite distances differs in certain ways from molecular diffusion. For instance, the assortment of eddy scales present, ranging from microscopic sizes to the size of the diffusion region itself, results in the largest mixing length being of the order of magnitude of the diffusion distance.

Thus the eddy diffusivity may be expected to be both time and distance dependent, approaching constancy at large values of x and t .

Agreement with Taylor's diffusion equations was obtained also in recent investigations by Keyes (D-9) and Deisler and Wilhelm (D-10).

(ii) Turbulent diffusion or dispersion largely created by boundary drag or shear flow.

Experimental determination of turbulent diffusion in shear flow is reported for circular conduits (D-6, D-14, D-15) and in two-dimensional channels (D-7, D-8).

In two papers Taylor (F-1 and D-3) has presented the equations for diffusion about a plane moving with the average flow velocity for laminar and turbulent flow through conduits.

In the case of laminar flow through a smooth tube the dispersion process is caused by the combination of radial molecular diffusion and convective longitudinal dispersion, a virtual diffusion coefficient is experimentally obtained, Taylor (F-1):

$$D_{\text{lam.}} = \frac{r^2 \bar{u}^2}{48 D_{\text{mol.}}} \quad [17]$$

In the case of turbulent flow through a conduit Taylor (D-3) obtains a theoretical expression for the dispersion coefficients due to convection and turbulent velocity components by the following assumptions:

1. Validity of the universal velocity distribution law:

$$\frac{U_0 - \bar{u}}{\sqrt{\tau_0/\rho}} = \text{fn}\left(\frac{r}{r_0}\right) \quad [18]$$

applicable for fully developed turbulent flow through either smooth or rough conduits.

2. Validity of the Reynold's analogy which states that the transfer of heat, mass and momentum by turbulence is analogous. Taylor points out that the Reynold's analogy is found to hold for shear generated turbulence, such as flow over a flat plate or in a conduit, but is not quite true for free turbulence, such as generated behind screens or obstacles to unconfined free flow. Reynold's analogy may be written as:

$$E = \frac{\tau}{\rho} \frac{\partial u}{\partial r} = \frac{N}{\rho} \frac{\partial c}{\partial r} \quad [19]$$

or

$$E = E$$

Taylor obtains the following values theoretically for turbulent dispersion coefficient:

Longitudinal convective dispersion,

$$E_1 = 10.06r_o \sqrt{\frac{u}{\rho}} \quad [20]$$

where $\sqrt{\frac{u}{\rho}}$ = fn. of R for smooth pipes and fn. of R and k for rough pipes.

Radial and longitudinal diffusion due to turbulent velocity components,

$$E_2 = 0.052r_o \sqrt{\frac{u}{\rho}} \quad [21]$$

Combined longitudinal diffusion due to convection and turbulent diffusion,

$$E = E_1 + E_2 = 10.1r_o \sqrt{\frac{u}{\rho}} \quad [22]$$

or in engineering units, for pipes,

$$E = 1.785dU\sqrt{f} \quad [23]$$

and for rivers or channels,

$$E = 14.28R^{3/2} \sqrt{2gS_e} \quad [24]$$

$$= 7.14RU\sqrt{f} \quad [24a]$$

d = pipe diameter = $2r_o$

where

R = hydraulic radius

u = mean flow velocity

f = friction factor (Darcy-Weisbach)

The values in the foregoing expressions apply only for perfectly straight and uniform conduits or channels and were experimentally verified by Taylor using test lengths of 3/8-inch diameter smooth and a 3/8-inch diameter roughened conduit, and was found to agree well with earlier experimental observations by others, e.g. Worsler (D-20). Tests with commercial distribution pipe lines or curved conduits indicate that the eddy diffusion coefficient is highly sensitive to the slightest deviation from the perfectly straight and uniform condition and may be two to three times the theoretical value where there is an average

amount of bends, change of shape, etc.

Data on diffusion coefficients from pipe tests in the field are contained in a thesis by Parker (E-14, pp. 4-22). The order of magnitude of E is 1 to 10 ft.²/sec. for a range of diameters up to 15 ft.

For a finite concentrated dosage the dispersion about a point moving with average velocity U is given by:

$$c = \frac{M}{A\sqrt{4\pi Et}} e^{-\frac{(x-Ut)^2}{4Et}} \quad [25]$$

where

M = weight of dose

(iii) Turbulent diffusion due to kinetic energy of diffusant: momentum heat, mass transfer in free submerged turbulent jets.

A review of the literature on turbulent diffusion would not be representative without a survey of turbulent jet diffusion. From the point of view of waste disposal, diffusion by submerged jets is an effective means of initial dilution. To establish the proper relationship between the degree of dilution that can be effected by the best use of the available kinetic energy of the diffusant inflow and the diffusion that can be effected by the turbulent kinetic energy of the receiving medium, it is desirable to include the following review of turbulent jet diffusion. For an extensive treatment see Pai (J-10).

In the case of a gaseous or liquid jet discharging into a stationary body of the same fluid, one is interested in the diffusion of the properties of momentum, heat and mass. A submerged jet issuing from a nozzle may be divided into two zones: (i) the zone of flow establishment, that is the region near the orifice; and (ii) the zone of established flow, i.e. the region beyond 6 or 7 orifice diameters. Most of the past research on jets has been performed with air or gases by means of the hot-wire anemometer. Lately (J-8) experiments with liquid jets yielded results in good agreement with those for gaseous jets.

A literature review reveals that Tollmien (J-1) in 1926 first established the distribution function of the temporal mean velocity in the flow field of a submerged jet; Corrsin (J-2) investigated the mechanics of turbulent fluctuations and this was further analyzed by Liepmann and Laufer (J-3). A general experimental and theoretical survey of the problem of jet diffusion is published by Rouse e.a. (J-4).

Studies relating the heat and mass transfer with momentum transfer were carried out by Lin (J-5), Hinze (J-6), Forstall and Shapiro (J-7) with gaseous jets and by Forstall and Gaylord (J-8) and Wille (J-9) with water jets.

The general diffusion equation, as applied to jets, may be stated as follows: the time rate of transfer of a property of the diffusing fluid through unit area of temporal mean stream surface is equal to the gradient of mean concentration normal to the stream surface times an appropriate turbulent diffusion coefficient.

The diffusion equation assumes the following forms according to the diffusing property in question:

$$\text{For momentum transfer: } \frac{\tau}{\rho} = \frac{\eta \partial u}{\rho \partial y} = -\epsilon \frac{\partial u}{\partial y} \quad [26]$$

$$\text{For heat transfer: } \frac{q}{\rho c_p} = -\frac{K_t}{\rho c_p} \frac{\partial \theta}{\partial y} \quad [27]$$

$$\text{For mass transfer: } N = -E \frac{\partial c}{\partial y} \quad [28]$$

The turbulent diffusion coefficients in the above equations involve the turbulent characteristics of the flow configuration and are not fluid properties as in the case of molecular diffusion. The properties of heat transfer and material transfer may be linked to that of momentum transfer by the turbulent Prandtl and the turbulent Schmidt Number respectively.

$$P_t = \frac{\eta^c}{K_t} = \frac{\epsilon}{E_{\text{heat}}} \quad [29]$$

$$S_t = \frac{\eta}{\rho D_t} = \frac{\epsilon}{D_t} = \frac{\epsilon}{E_{\text{mass}}} \quad [30]$$

It has been found by many experimenters that these two numbers are approximately equal to 0.7 for a large range of gases, signifying that heat and material diffuse more rapidly than momentum across the mean stream surfaces. The explanation for this behavior is that clear fluid is entrained from the environs outside the jet towards the centerline. The momentum is easily transferred to the fluid being entrained, but the material and heat content of the jet is diminished faster along the center line since the entrained water displaces packets of fluid with higher heat and material concentration which tend to feather off radially to the periphery of the jet. As a net result material and heat diffuse faster than momentum.

Since viscous and inertia forces predominate, the studies may be made non-dimensional by expressing all variables in terms of a Reynolds Number and an initial density and temperature difference. The dependent variable is a length parameter expressed in terms of nozzle diameters. The density difference affects largely the scale of the turbulence and not the free body motion in the zone near the nozzle.

Thus one can determine theoretically a non-dimensional velocity, temperature or concentration of material distribution at any distance from the orifice in the region of established flow. The momentum across a section normal to the jet axis has to be a constant equal to $\rho U_o^2 A_o$. This leads to the hyperbolic center line velocity distribution:

$$\frac{u_o}{U_o} = \frac{D_o}{2Cx} \quad [31]$$

The radial velocity distribution within the jet follows a Gaussian curve, as first suggested by Tollmien and with the values of the constant proposed by Rouse (J-11), assumes the form:

$$\frac{u}{U_o} = \frac{D_o}{2Cx} \exp. \left[-\frac{1}{2C^2} \frac{r^2}{x^2} \right] \quad [32]$$

where

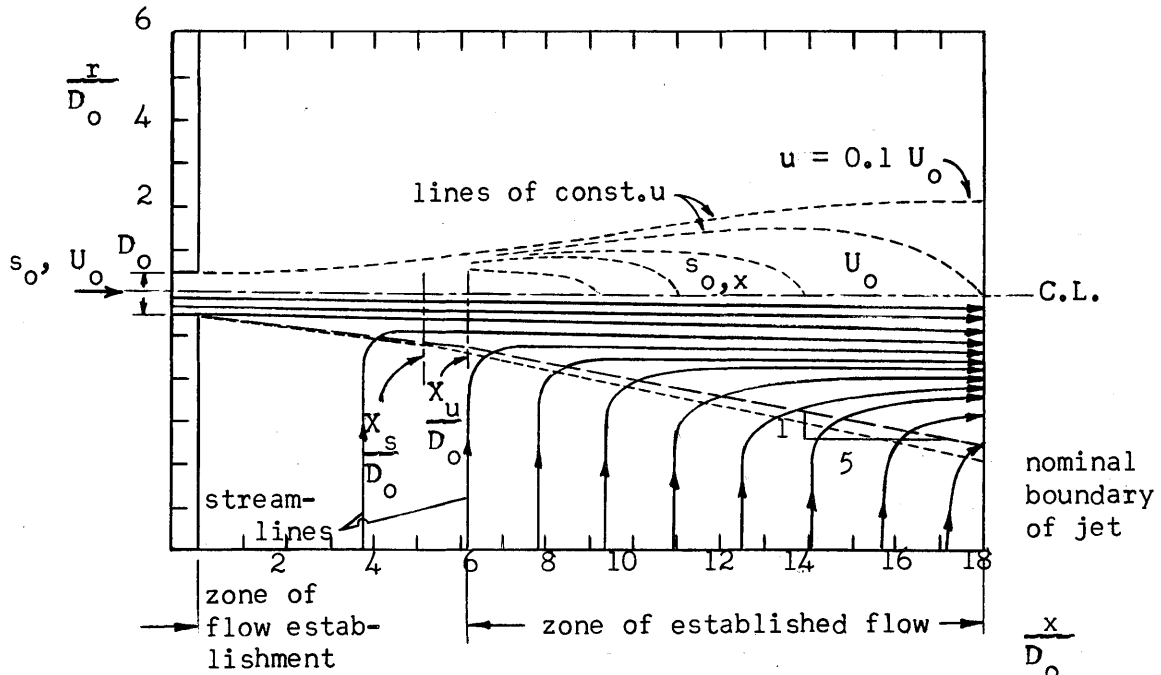
$$\frac{1}{2C} = 6.2$$

or

$$u = \frac{U_o D_o}{x} e^{-(1.82 - 76 r^2/x^2)} \quad [32a]$$

For heat or mass transfer one replaces C by:

$$\left. \begin{aligned} C_\theta &= \sqrt{P_t C} \\ C_g &= \sqrt{S_t C} \end{aligned} \right\} \approx 0.85 \text{ to } 0.95 C$$

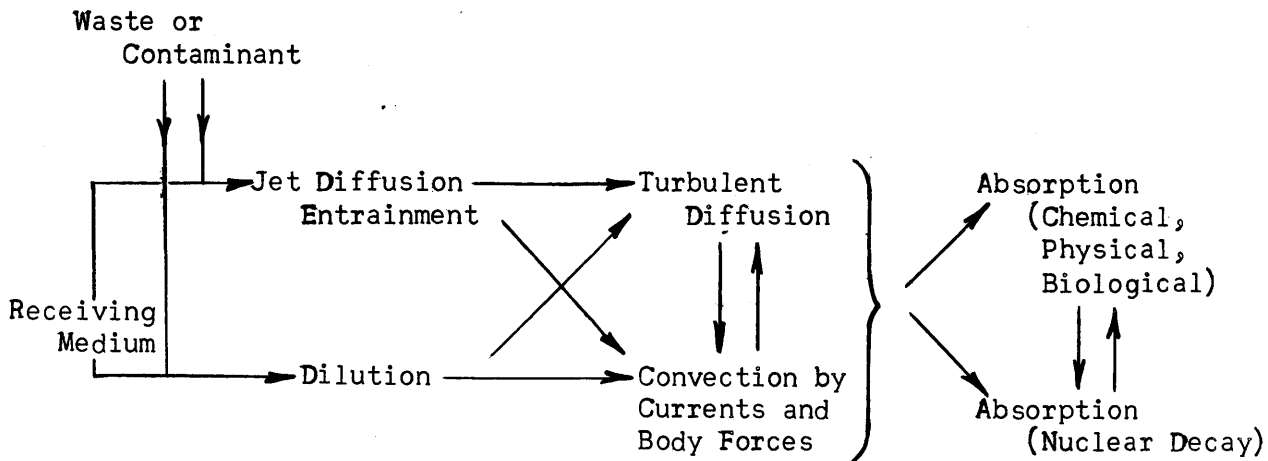


Definition Sketch for Circular Submerged Jet. (after Rouse, J-4).

C. Diffusion with or without convection action of body forces; non-conservative concentration of diffusing properties.

Associated with the disposal of industrial wastes in a natural receiving body of water are factors which affect concentration distribution as a function of time other than the initial dilution by entrainment and the subsequent diffusion by turbulence. The most important of these factors are: (i) the action of gravity on fluid masses of varying density and (ii) the absorption of some of the contaminant constituents. The absorption may be due to physical or chemical processes (such as flocculation and settling, pH variation in the receiving medium, aeration and organic decomposition) or may be the result of biological digestion, or due to decay of radioactive isotope contaminants.

The following diagram serves to clarify the terminology:



Processes involved in ultimate disposal of contaminant in natural body of water

A review of mathematical solutions for special cases follows.

(i) Effects of body forces due to density variation.

By means of the Furth transformation (A-5) a certain class of time-dependent diffusion where there is an external force causing a convective velocity in the direction of diffusion may be solved. In the one-dimensional convection equation

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} - v \frac{\partial c}{\partial x} \quad [15a]$$

where

v = fall velocity under external force

by the transformation

$$c = c^* e^{\left[\frac{v}{2D}(x - x_0) - \frac{v^2 t}{4D} \right]} \quad [33]$$

one obtains:

$$\frac{\partial c^*}{\partial t} = D \frac{\partial^2 c^*}{\partial x^2} \quad [34]$$

The solution for the one-dimensional semi-infinite non-steady state sedimentation process is given in Ref. (A-5) after Furth. In this connection see also Mason and Weaver (A-14).

(ii) Absorption including radioactive decay.

Extensive treatments of diffusion and chemical reaction processes are available in standard references on diffusion (Barrer A-4, Jost A-5). Parker (E-14) presents the solution (see below) for the two-dimensional concentration distribution as a function of radial distance and time, resulting from dosing a detention reservoir with radioactive contaminant. A solution is also given for the steady state distribution of concentration due to a constant inflow of radioactive contaminant into a flowing stream or idealized reach in a reservoir.

Diffusion without convection.

For the two-dimensional diffusion of a mass M per unit depth of non-decaying contaminant, initially introduced in a line source extending vertically over the entire depth of the receiving reservoir, assumed to be uniform in depth and of great extent.

The solution of the differential equation

$$\frac{\partial c}{\partial t} = E \frac{\partial^2 c}{\partial x^2} \quad [2]$$

is Eqn. [45], $n = 2$,

$$c = \frac{M}{4\pi Et} e^{-\frac{x^2 + y^2}{4Et}} \quad [36]$$

Where the contaminant is a radioactive isotope and may, in addition, be physically or chemically non-conservative, two additional factors besides turbulent diffusion enter into consideration: (i) nuclear decay and (ii) absorption due to physical, chemical or biological causes.

Without any turbulent diffusion the concentration of isotope in a dosage at time t would be given by:

$$c = c_0 e^{-(K + \gamma)t}$$

where

c_0 = concentration at $t = 0$

K = nuclear decay constant

γ = absorption constant

The resulting solution for the concentration becomes:

$$c = \frac{M}{4\pi Et} e^{-\frac{x^2 + y^2}{4Et} - (K + \gamma)t} \quad [38]$$

Diffusion with convection, nuclear decay and absorption.

For an ideal reach, the continuity equation for steady state one-dimensional diffusion with convection and nuclear decay gives:

$$v \frac{dc}{dx} = E \frac{d^2c}{dx^2} - Kc \quad [39]$$

The differential equation is of the form

$$(D^2 - \frac{u}{E} D - \frac{K}{E})c = 0$$

which has the solution:

$$c = b_1 e^{\left[\frac{u}{E} + R\right] \frac{x}{2}} + b_2 e^{\left[\frac{u}{E} - R\right] \frac{x}{2}} \quad [40]$$

where

$$R = \sqrt{\frac{u^2}{E^2} + \frac{4K}{E}}$$

Parker obtained the exact solution for Eqn. [39] for steady state, as well as an approximate solution more suited for calculations. A solution was also obtained by a step method for the time that the steady state condition takes to build up.

The equation for the unsteady state problem was solved by a step method which checked with the exact solution and was substantiated by field observations.

For the case of dispersion in streams and aqueducts, the solution for Eqn. [39] is:

$$c = \frac{M}{A\sqrt{4\pi Et}} e^{-\frac{(x - ut)^2}{4Et} - (K + \gamma)t} \quad [41]$$

Representative values of the coefficient are:

$$K = .0495/\text{day}$$

$$\gamma = .0326/\text{day}$$

$$E = 10 \text{ to } 100 \text{ ft.}^2/\text{sec.}$$

D. (i) Mechanics of waste water disposal by dilution and diffusion into a natural receiving body of water.

The disposal of liquid wastes by dilution may be classified under the categories of ocean disposal, disposal in estuaries, disposal in rivers and lakes and contamination of watersheds. The process of diffusion in a practical situation may involve one or more of the idealized cases of diffusion (with or without convection, with or without property conservation) considered in the previous sections.

Ocean Disposal. Rawn and Palmer (E-1) presented empirical methods for designing ocean sewage-discharge outfalls taking into account factors such as initial jet mixing, rise and spread of the pollution field and its dispersion by winds and currents. Warren (E-2) and Pearson (E-3) have also treated the subject of ocean disposal. Morton, Taylor and Turner (E-4) treat cases of turbulent diffusion and gravitational convection in the atmosphere (analogous to a rising plume of effluent from an undersea sewage outfall). Expressions are obtained for the rise of the plume from a maintained source for constant density, as well as for linear variation of density of receiving fluid. The assumption is made that the turbulent entrainment is proportional to the vertical velocity. Munk (E-5) and coworkers report on the measurement of horizontal and vertical eddy diffusivities in the Bikini lagoon. Eddy diffusivities were obtained from the spread of dye markers before and from measurements of radioactivity after the atomic bomb test. E_x was obtained from the source-function solution and E_y from differentiation of \bar{C} concentration with respect to distance and depth.

$$E_y = \frac{u \frac{\partial s}{\partial x}}{\frac{\partial^2 s}{\partial y^2}}$$

Disposal in Estuaries. Rouse (E-6) treats the subject of gravitational convection and diffusion in two-dimensional turbulent flow, (heated air currents rising in an atmospheric wind) analogous to the discharge of industrial wastes into water of higher density. Ketchum (E-7) surveys the self-cleaning action of tidal estuaries. Diachishin and coworkers (E-8) report on the subject of sewage disposal in estuaries. Research on the basic problem of tidal mixing has been pioneered by Arons and Stommel (E-9) and Stommel (E-10).

Disposal in Rivers and Lakes, Contamination of Watersheds, Reservoirs, Aqueducts. The contamination of water supplies by radioactive precepitation on watersheds and disposal of industrial and radioactive wastes, has been thoroughly investigated - Parker (E-14), Gurnham (E-11), Straub (E-12).

Parker presents methods (see Section C) for routing of the contaminant concentration through natural detention bodies, and for the diffusion, absorption and decay of contaminant with flow in streams and aqueducts.

General Background.

Texts on industrial waste treatment (Gurnham E-11) yield statistics on types of industrial wastes and pretreatment processes prior to final disposal. Sanitary standards for the ultimate disposal of wastes and specifications for allowable pollution levels for water supplies as prescribed by state and federal health authorities, constitute a subject of its own, apart from the subject of this thesis.

The application of turbulent diffusion theory to waste water disposal starts with the assumption that certain allowable ultimate concentration limits have been prescribed and that the waste influent concentration and flow rates to be designed for, are specified.

(ii) Determinations of diffusion coefficients from concentration fields or vice versa.

Under Theoretical Considerations (Chapter III) are presented basic methods for determining diffusion coefficients, and for obtaining the concentration field for given boundary conditions and values of diffusion coefficients. Numerical values of vertical and lateral dynamic eddy diffusivities and eddy viscosities for oceans and ocean currents are given by Sverdrup, Johnson, and Fleming (E-13). Typical values for the lateral eddy diffusivity are 10^7 to 10^8 times greater than representative values for the vertical eddy diffusivity, e.g.

$$\rho E_y = 1 \text{ to } 320 \text{ g/cm/sec.}, \quad \rho E_x = 4 \times 10^7 \text{ to } 4 \times 10^8 \text{ g/cm/sec.},$$
$$\rho \epsilon_y = 10 \text{ to } 7500 \text{ g/cm/sec.}, \quad \rho \epsilon_x = 7 \times 10^7 \text{ g/cm/sec.}$$

For the oceans, as a rule the vertical eddy diffusivity is found to be less than the vertical eddy viscosity, by a factor of about 10. In Ref. (E-13) a chapter is devoted to distribution of scalar and vector fields of property concentration in a fluid body, e.g. the ocean. In addition, the most general forms of the differential equations for diffusion with convection, for both conservative and non-conservative diffusant properties, are given. It is indicated how these equations may be simplified in accordance with local conditions, yielding methods for determining turbulent flow properties (e.g. eddy viscosity or eddy diffusivity) by successive differentiation of the measured diffusant property concentration (velocity or salinity) fields.

Proudman (B-3) deals with mixing and turbulent eddy diffusion for several cases: steady-state distribution due to mixing along a current, time-dependent mixing in one and three dimensions in a constant uniform current, and gives mathematical solutions to certain simplified forms of the differential equation.

III THEORETICAL CONSIDERATIONS

A. Review of classical diffusion theory and methods for determining diffusion coefficients.

1. Time-dependent diffusion.

For solving uniquely the differential equation of diffusion (heat or mass transfer) through a medium with diffusivity a constant, certain boundary conditions have to be specified. It is convenient to work with a medium of semi-infinite extent since one less boundary condition needs to be fulfilled.

For the semi-infinite medium it is sufficient to specify the initial distribution of diffusant and the boundary condition at the source which is:

(1) For the one-dimensional case, a plane source with either (i) a finite quantity (mass) of diffusant, or a prescribed function with time of (ii) concentration potential, or (iii) strength of flux; in each case uniformly distributed throughout the plane of the source.

(2) For the two-dimensional case, a line source with uniformly distributed (i) finite quantity, or a prescribed function of (ii) concentration potential or (iii) strength of flux.

(3) For the three-dimensional case, a point source of (i) specified quantity, or prescribed time function of (ii) concentration potential, or (iii) strength of flux.

The three boundary value conditions: (i) source function, (ii) potential function, (iii) flux function, are all time-dependent functions.

In the case of constant flux of diffusant into a two or three-dimensional receiving medium, the ultimate distribution ($t \rightarrow \infty$) is approached asymptotically (quasi-steady state). In the one-dimensional case no ultimate distribution is approached - for constant inflow of diffusant a given concentration will continuously progress away from the source with speed proportional to $\frac{1}{\sqrt{t}}$.

(i) Source function.

A finite quantity of diffusant is initially concentrated at a source situated infinitesimally close to the boundary of the semi-infinite diffusing medium. The boundary is insulated and the diffusant is at liberty to distribute by diffusion within the medium.

$$\int_R c dV = M, \left(\frac{\partial c}{\partial x}\right)_{0, t} = 0$$

Consider the one-dimensional diffusion equation for constant diffusion coefficient (Fick's Second Law, see p. 6):

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} \quad [2]$$

To obtain the most simple solution one determines the values of the arbitrary functions of x and t in the trial solution (A-5):

$$c_{x,t} = g(x) h(t) e^{-\frac{Kx^2}{t}} \quad [41]$$

that satisfy the diffusion equation [2] and the additional requirement that c for $t = 0$ at $x = 0$ is finite. The functions that satisfy these conditions are:

$$g(x) = \text{const.} = \alpha$$

$$h(t) = \frac{1}{\sqrt{t}}$$

$$K = \frac{1}{4D}$$

thus:

$$c_{x,t} = \frac{\alpha}{\sqrt{t}} e^{-\frac{x^2}{4Dt}} \quad [42]$$

Consider an infinite tube of unit cross-section containing the diffusion medium. The amount of diffusing substance (diffusant) contained in an element of length Δx is then:

$$\Delta m = c \cdot \Delta x \cdot 1$$

Integrating from $-\infty$ to $+\infty$ yields the total amount of diffusing substances present

$$\int_{x=-\infty}^{x=+\infty} dm = \int_{-\infty}^{\infty} c dx = \int_{-\infty}^{\infty} \frac{\alpha}{\sqrt{t}} e^{-\frac{x^2}{4Dt}} dx$$

letting $\eta^2 = \frac{x^2}{4Dt}$

$$= 2\sqrt{Dt} \frac{\alpha}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta$$

$$= 2\sqrt{Dt} \alpha \sqrt{\pi} = \text{const.} = M \quad [43]$$

[Where M = total amount (mass) of diffusing substance present (which had been concentrated at $t = 0$ over the cross-section $x = 0$) and has to be a constant by the law of conservation of mass.]

Hence

$$\alpha = \frac{M}{2\sqrt{\pi Dt}} \quad \text{and [42] becomes:}$$

$$c_{x,t} = \frac{M}{2\sqrt{\pi Dt}} e^{-\frac{x^2}{4Dt}} \quad [44]$$

The resulting distribution is the familiar Gauss error curve¹

In a similar way (A-7) the distributions of concentration can be expressed for 2, 3 or n dimensions as:

$$c_{r,t} = \frac{\text{Const.} \cdot e^{-\frac{x^2}{4Dt}}}{t^{n/2}} \quad n = 0, \text{ integer } 1, 2 \text{ or } 3$$

$$= \frac{M}{(4\pi Dt)^{n/2}} e^{-\frac{r^2}{4Dt}} \quad [45]$$

Curves for the source-functions are given for one, two and three dimensional cases in Ref. A-6, (pp. 47, 48 of App.)

Determination of D from mean-square displacement of tracer dosage.

One, two or three-dimensional cases.

Consider Eqn. [44] and footnote following;

$$c_{x,t} = \frac{M}{2\sqrt{\pi Dt}} e^{-\frac{x^2}{4Dt}} = p(x) \quad [44]$$

The probability for diffusing particle to be between places x and x + dx at time t is:

$$p_x = \frac{p(x) dx}{\int_{-\infty}^{\infty} p(x) dx} = \frac{\frac{M}{2\sqrt{\pi Dt}} e^{-\frac{x^2}{4Dt}} dx}{\frac{M}{2\sqrt{\pi Dt}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4Dt}} dx}$$

1

The concentration distribution curve represents at the same time to an appropriate scale the probability of finding a particular particle of diffusing substance, initially released at x = 0, at an ensuing time t in the interval between two cross-sectional planes unit distance apart, and situated at a distance x from the origin. Since the total area under the curve is an invariant with time it may be considered to represent unity probability of a particular diffusing particle being anywhere within the tube. The area under the distribution curve between x and x + Δx at a given value of t divided by the total area thus represents the probability of the particle being between these planes at time t. This leads to a useful method of determining D from measuring the mean square displacement versus time of a tracer substance deposited in a fluid body, which will be discussed later.

The denominator simplifies to M, thus

$$p_x = \frac{1}{2\sqrt{\pi Dt}} e^{-\frac{x^2}{4Dt}} dx \quad [46]$$

The mean square displacement² may be defined as the summation of the displacement squared times the fractions of diffusing particles within that displacement interval. This fraction is the same as the probability of one particle to be within a certain displacement interval and is thus equal to p_x .

Hence:

$$\begin{aligned} \overline{\Delta x^2} &= \sum_{x=-\infty}^{\infty} x^2 p_x \\ &= \int_{-\infty}^{\infty} \frac{x^2}{2Dt\pi} e^{-\frac{x^2}{4Dt}} dx \end{aligned}$$

Let $\frac{x^2}{4Dt} = \eta^2$

$$= \frac{4Dt}{\sqrt{\pi}} \int_{-\infty}^{\infty} \eta e^{-\eta^2} d\eta \quad [47]$$

= 2Dt, for one-dimensional diffusion.

For diffusion in n dimensions a similar analysis yields:

$$\overline{\Delta x^2} = 2nDt \quad [47a]$$

The practical advantage of this method lies therein that D can be determined for diffusion in one, two or three coordinate directions by means of very simple and compact apparatus: a source of diffusing substance, and a concentration measuring probe. By injecting a quantity of substance into the fluid body and determining the mean square displacement of the concentration after time t, the diffusion coefficient D may be obtained by substitution in Eqn. [47a]. (Where convection exists, a moving coordinate system should be used. The method is not applicable where convection is non-uniform).

The value of $\frac{c}{c_0}$ at $x = \sqrt{\Delta x^2}$ for a Gaussian distribution is 0.6063.

2

Equivalent to standard deviation or radius of gyration of area under curve.

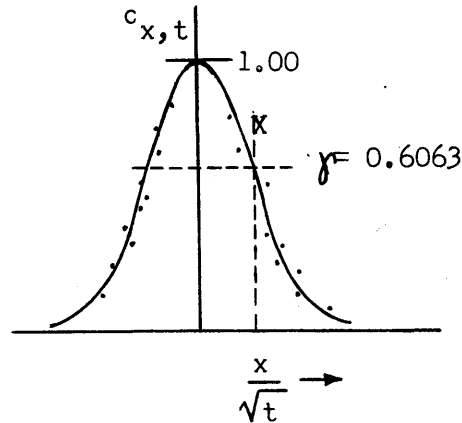
If the distribution curve is plotted as $\frac{c}{c_0}$ versus $\frac{x}{\sqrt{t}}$ for various values of t , the intersection of the Gaussian curve with the line $\frac{c}{c_0} = 0.6063$

will yield the value $\sqrt{\frac{\Delta x^2}{t}} = X$

Hence:

$$D = \frac{\overline{\Delta x^2}}{2nt} = \frac{X^2}{2n} \quad [48]$$

where $n =$ number of dimensions.



(ii) Constant potential function - one-dimensional diffusion.

Consider a one-dimensional³ diffusion process such as would occur in a long infinite tube of unit cross-sectional area containing a diffusing medium and a diffusing substance.

Suppose initially that the concentration of diffusing substance is $2C_0$

³ For the two and three dimensional cases a similar analogy applies. By considering in the 2-D case two semi-infinite slabs $0 < z < 1$, isolated from each other along the surface $y = 0$, except for a common line contact at $x = 0$, where the diffusing substance can pass freely from the one half to the other, one can postulate the following condition:

Suppose initially that the left semi-infinite slab contains a concentration $2C_0$ of diffusing substance and that the right semi-infinite slab contains concentration zero. One can say that the concentration in the right half at a radius r from the origin is the sum of the concentrations caused by each source element in the left half, all the diffusion flow occurring via the line contact at the origin. The summation may be performed exactly as for the one-dimensional case except that it becomes a surface integral of the source function instead of a line integral. In the three-dimensional case the concept is extended to a volume integral of the source function throughout a left hand semi-infinite space and a right hand semi-infinite space that has only one common point at the origin.

in the left-hand portion $-\infty < x < 0$ and that the concentration is zero in the right hand portion $0 < x < \infty$. Consider influence of the diffusion of the substance contained in an element of length Δx , situated immediately to the left of $x = 0$ ($-\Delta x < x < 0$) on the concentration of substance at a point $x = \xi$ to the right of the element after a time t has elapsed. The element $-\Delta x < x < 0$ constitutes a source containing at $t = 0$ a definite amount of substance $\Delta M = 2c_0 \cdot 1 \cdot \Delta x$ which is diffused thereafter according to the source function in both positive and negative x -directions.

Since the diffusion equation is linear, one obtains by summation of the concentration increments at $x = \xi$, Δc due to all source elements Δx between $x = 0$ and $x = -\infty$, the concentration at $x = \xi$ at time t in the right-hand portion as:

$$c = \int_{x=-\infty}^0 dc = \int_{-\infty}^0 \frac{2c_0}{2\sqrt{\pi Dt}} e^{-\frac{(\xi-x)^2}{4Dt}} dx$$

Let $\eta = \frac{\xi-x}{2\sqrt{Dt}}$, hence

$$c_\xi = \frac{2c_0}{\sqrt{\pi}} \int_{\xi}^{\infty} e^{-\eta^2} d\eta$$

Let $\eta_0 = \frac{\xi}{2\sqrt{Dt}}$, hence

$$c = c_0 [\text{erf}(\infty) - \text{erf} \eta_0],$$

where $\text{erf} \eta_0 = \frac{2}{\sqrt{\pi}} \int_0^{\eta_0} e^{-\eta^2} d\eta$, $\text{erf} \infty = 1$.

Writing x for ξ ,

$$c_x = c_0 \left(1 - \text{erf} \frac{x}{2\sqrt{Dt}}\right) \quad [49]$$

(a) Determination of D from concentration versus x/\sqrt{t} for one-dimensional case (for constant D).

From Eqn. [49] it is seen that D could be determined from any experimental value of concentration versus x and t . The concentration at $x = 0$ is equal to c_0 for all values of $t > 0$. Since $\left. \frac{\partial \text{erf} \eta}{\partial \eta} \right|_{\eta=0} = \frac{2}{\sqrt{\pi}}$ the concentration gradient at $x = 0$ is

$$\left. \frac{\partial c}{\partial x} \right|_{x=0} = \frac{d}{d\eta} [c_0 (1 - \text{erf} \eta)] \frac{\partial \eta}{\partial x},$$

where $\eta = \frac{x}{2\sqrt{Dt}}$

$$= -\frac{c_0}{\sqrt{\pi Dt}} \quad [50]$$

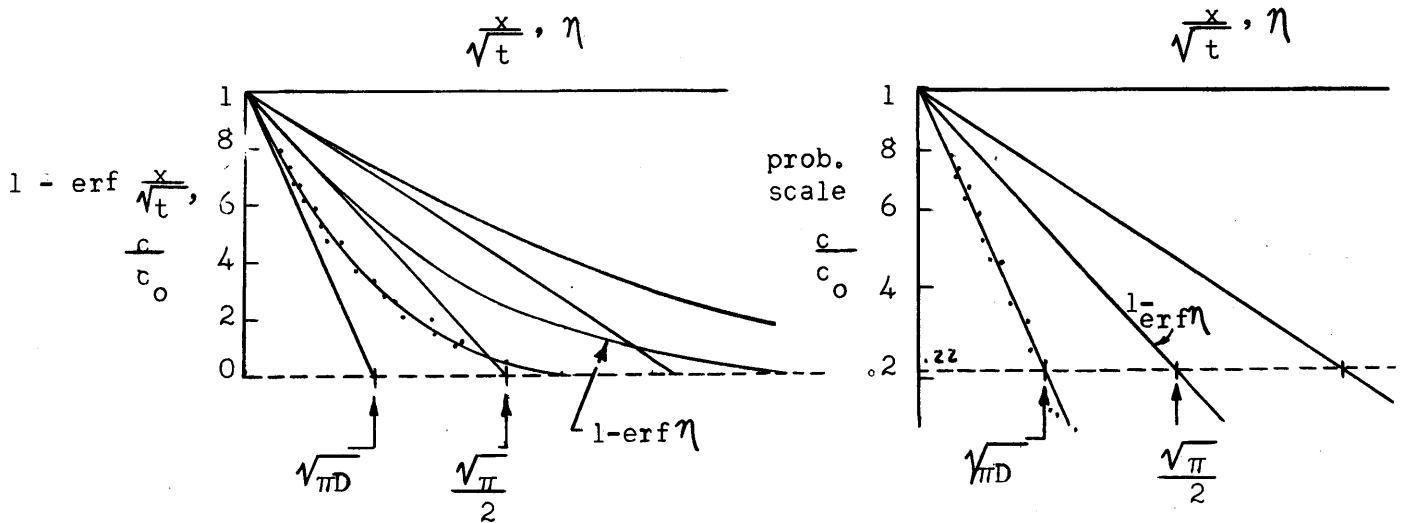
Thus, by taking concentration measurements at a small distance x , from the interface one can obtain the gradient at $x = 0$ and hence D , by substituting in

Eqn. [50]. This method is liable to yield erroneous results and a better method is the following, where D is determined from the entire range of concentration distributions.

Rewriting Eqn. [49] as:

$$1 - \frac{c_{x,t}}{c_0} = \text{erf} \frac{x}{2\sqrt{Dt}} \quad [49a]$$

If $\text{erf} \eta$ is plotted on a probability scale versus η on an arithmetical scale, the slope of the straight line obtained is equal to $\frac{2}{\sqrt{\pi}}$. Thus, plotting $1 - \frac{c_{x,t}}{c_0}$ on a probability scale versus $\frac{x}{\sqrt{t}}$ on arithmetical, the points should fall along a straight line of slope $\frac{1}{\sqrt{\pi D}}$.



Thus, where D is a constant, it may be obtained simply from a graphical curve-fitting plot of the experimental concentration curves for various stations and times.

(b) Determination of variable D by differentiation of concentration versus x, t .

For D variable (time or distance dependent or both) the following method can be adopted provided that sufficient data are obtained to reconstruct the entire diffusion process, and that the data are accurate enough to warrant first order differentiation.

Consider Fick's Second Law for D a function of x, t , where $D = D(x, t)$. Integrating for $t = \text{constant}$:

at $x = \infty$

$$\int_0^x \frac{\partial c}{\partial t} dx = D \frac{\partial c}{\partial x} + \text{constant}$$

$$\int_0^{\infty} \frac{\partial c}{\partial t} dx = \left[D \frac{\partial c}{\partial x} \right]_{x=\infty} + \text{constant}$$

and since $\frac{\partial c}{\partial x} \rightarrow 0$ as $x \rightarrow \infty$, the constant is equal to the left hand side of the preceding equation.

Hence,

$$\int_0^x \frac{\partial c}{\partial t} dx = \left(D \frac{\partial c}{\partial x} \right) + \int_0^{\infty} \frac{\partial c}{\partial t} dx.$$

Simplifying the notation, the equation reduces to:

$$D_{x,t} \frac{\partial c}{\partial x} = I_x - I_{\infty}$$

$$D_{x,t} = \frac{I_x - I_{\infty}}{\left(\frac{\partial c}{\partial x} \right)_x} \quad [52]$$

This method is reliable in cases where I_{∞} can be determined accurately and where the difference $I_x - I_{\infty}$ is not less than about one tenth of I_{∞} .

(c) Determination of D from quantity of tracer substance diffused after a time interval. One-dimensional case (constant D).

In practice, determinations of D are often made by measuring the total quantity (mass) of substance diffused into a control volume after a time t. If the substance is a diffusing salt in a liquid medium, the amount of substance diffused may be accurately weighed after evaporating the liquid in a measured sample or by conductivity measurements.

By differentiation of Eqn. [49] at $x = 0$, we obtain:

$$\left. \frac{\partial c}{\partial x} \right|_{x=0} = - \frac{c_0}{\sqrt{\pi D t}} \quad [50]$$

But by Fick's First Law of Diffusion:

$$\begin{aligned} N &= - D \frac{\partial c}{\partial x} \\ &= c_0 \sqrt{\frac{D}{\pi t}} \end{aligned} \quad [53]$$

And⁴

$$S = A \int_0^t N dt = c_0 \sqrt{\frac{D}{\pi}} A \int_0^t t^{-1/2} dt$$

$$= 2c_0 A \sqrt{\frac{Dt}{\pi}}$$

Hence,

$$D = \frac{S^2 \pi}{A^2 (2c_0)^2 t} \quad [54]$$

Where,

$2c_0$ = initial concentration

A = cross sectional area

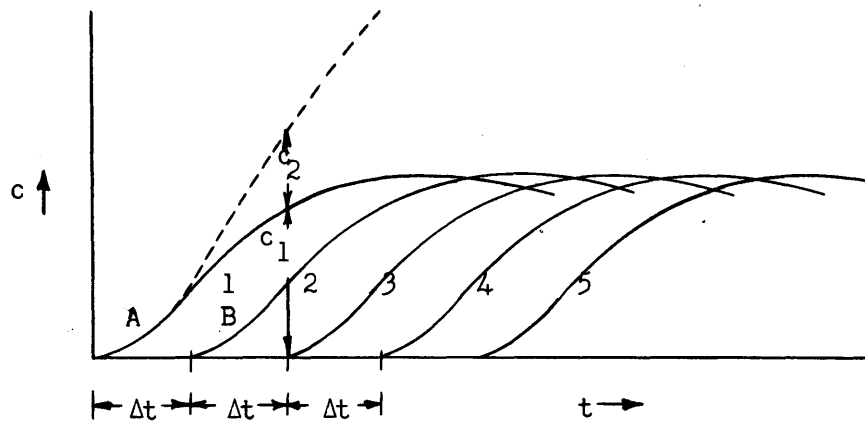
S = total amount of substance diffused after time t

(iii) Constant flux function.

This represents cases where a uniformly distributed constant rate of influx is maintained throughout the source. In the two-dimensional case, one may set up accordingly a boundary condition of constant uniformly distributed flux of diffusing substance from a line source into the diffusing medium.

Consider the effect of a source of diffusing substance on the concentration at a point distance r from the source. From $t = 0$ to $t = \Delta t$ a finite quantity of substance ΔS is diffused from unit length of source where

⁴ S may also be expressed as $\int c_t dV = \bar{c}_t \frac{AL}{2}$, where \bar{c}_t is the mean concentration in the volume at time t . In practice two methods may be used for determining D . By the first method the substance is allowed to diffuse for a time t (taking precaution that the concentration at the finite boundary remains negligible), after which the source is cut off from the control volume V and the substance allowed to continue diffusing. After a sufficient time interval the concentration \bar{c}_t may be obtained by averaging a sufficient number of evenly distributed samples. This method would be useful where instantaneously-recording concentration probes are not practical. By the second method, the concentrations are taken at frequent time intervals. S may be obtained by evaluating $\int A c_t dx$ for various times.



$$\Delta S = q\Delta t \quad [55]$$

which causes the above concentration time curve at x , curve A. If the same amount of substance is introduced during time interval $t = \Delta t$ to $t = 2\Delta t$, a curve follows that is displaced to the right by Δt , curve B. Adding the ordinates of the successive curves at intervals Δt apart will give the curve (shown dashed) for the concentration at x versus time. This is the same as adding the ordinates of the single curve A at Δt intervals from $t = t$ to $t = 0$. Hence, the constant flux function is obtained by integrating the source function with respect to time.

In the limit as $\Delta t \rightarrow 0$, there results:

$$c = \int_{t=0}^{t=t} dc$$

From Eqn. [45] with $n = 2$ for two-dimensional case:

$$\Delta c = q \frac{\Delta t}{4\pi D t} e^{-\frac{r^2}{4Dt}} \quad [56]$$

Integrating with respect to time:

$$c = q \int_0^t \frac{1}{4\pi D t} e^{-\frac{r^2}{4Dt}} dt$$

Let

$$u = \frac{r^2}{4Dt}$$

Hence,

$$c = \frac{q}{4D\pi} \int_{\frac{r^2}{4Dt}}^{\infty} \frac{e^{-u}}{u} du$$

By definition

$$-\text{Ei} \left(-\frac{x^2}{4Dt} \right) = \int_{\frac{x^2}{4Dt}}^{\infty} \frac{e^{-u}}{u} du, \quad 0 < x < \infty,$$

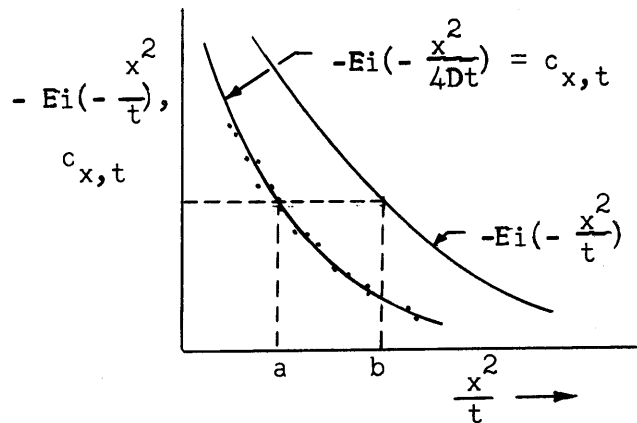
where Ei is the "exponential integral", hence:

$$c = -\frac{q}{4\pi D} \text{Ei} \left(-\frac{x^2}{4Dt} \right) \quad [57]$$

Determination of D from concentration versus x/\sqrt{t} for two-dimensional case.

D can thus be obtained for this case by means of a plot of c and

$-\text{Ei} \left(-\frac{x^2}{t} \right)$ versus $\frac{x^2}{t}$.



$$a \frac{x^2}{4Dt} = b \frac{x^2}{t}$$

Hence,

$$D = \frac{a}{4b} \quad [58]$$

It is seen from the above review of the classical diffusion theory that one has for each of the three-dimensional conditions three simplified boundary value problems available. Five of the nine possible combinations were shown to yield simple mathematical models for evaluation of diffusion coefficients.

The choice from the available methods for evaluating D would be determined by individual requirements, experimental feasibility and ease of computation.

2. Steady state diffusion and convection, determination of D for constant diffusant inflow into a constant current.

The one-dimensional steady state diffusion and convection equation (where $\frac{dc}{dt} = 0$ and $-V$ is the convective velocity) is given by:

$$Vc = D \frac{dc}{dx} \quad [59]$$

By integration, for constant D:

$$\ln \frac{c}{c_0} = - \frac{Vx}{D} \quad [60]$$

This relationship may be used for the evaluation of D from the quasi-steady state distribution of concentration with distance upstream from the diffusant source.

By plotting $\log \frac{c}{c_0}$ versus x and drawing a smooth curve can be closely represented by a straight line, it is seen that the slope of the line equals $-\frac{V}{2.3D}$, hence:

$$D = - \frac{VX}{2.3} \quad [60a]$$

where X is the interval of x , corresponding to a log cycle (ten-fold variation in concentration). If the curve deviates from a straight line, then D is a function of x and the integration yields:

$$\ln \frac{c}{c_0} = - \int \frac{Vdx}{D} \quad [60b]$$

For example, let

$$D = b + \frac{a}{x} \quad [60c]$$

the resulting distribution is

$$\frac{c}{c_0} = e^{-\frac{Vx}{b} \left(1 + \frac{ax}{b}\right)^{\frac{aV}{b^2}}} \quad [60d]$$

which reduces for $b = 0$, hence $D = \frac{a}{x}$, to:

$$\frac{c}{c_0} = e^{-\frac{Vx^2}{2a}}, \quad \text{the Gaussian Distribution} \quad [60e]$$

a and b in Eqn. [60c] may be readily solved from Eqn. [60d] by simultaneous equations for cases where a solution exists.

Time required for the establishment of steady-state condition.

It is difficult to tell when sufficient time has elapsed for the diffusion-convection process to have approached the quasi-steady state condition. This may result in doubt later on as to the validity of data and may cause errors when data

for widely differing conditions are compared where in some cases the steady-state may not have been attained.

The equations for the time-dependent concentration growth towards the steady-state have been formally derived by Mason and Weaver (A-14) and independently by Furth (A-15). The time τ taken to establish a concentration distribution that may be assumed for all practical purposes to be identical with the distribution at $t = \infty$ is derived by Weaver (A-16) as equal to $\frac{2H}{V}$, where

$$H = \frac{1}{c_0} \int_0^{\infty} c dx$$

hence

$$\tau = \frac{2}{V} \int_0^{\infty} \frac{c}{c_0} dx = \frac{2D}{V^2} \quad [65]$$

Since D is not yet known until the quasi-steady state is reached, the evaluation of H can be made from the concentration distribution taken when the concentration change at an index point becomes negligible. The experiment can then be continued until the time $t = \tau$ is reached, whereupon the final concentration distribution measurements are made.

B. Review of turbulent diffusion theory and special case of classical diffusion theory where diffusion coefficient is concentration-dependent.

1. Summary of Approximate Turbulent Diffusion Theory due to Goldstein.

Consider a turbulent field where the fluid particles move with finite jumps of equal lengths at constant speed V . Consider further that the motion is in one dimension only. If for a typical particle the probability p of two successive jumps to occur to the left is equal to the probability p' of two successive jumps to occur to the right, then the turbulence is called isotropic. If the probability q of occurrence of one jump to the right followed by one jump to the left is equal to the probability p of occurrence of two successive jumps to the right, the correlation is said to be zero.

Relationships such as these lead to difference equations resulting after considerable manipulation in the Goldstein turbulent diffusion equation:

$$\frac{\partial^2 c}{\partial t^2} + \frac{V^2}{D} \frac{\partial c}{\partial t} = V^2 \frac{\partial^2 c}{\partial x^2} \quad [A-1]$$

(hence by dividing through by V^2 and then letting $V \rightarrow \infty$ one obtains the classical diffusion equation [Eqn. 2]). (On the other hand, for $D \rightarrow \infty$ i.e. no resistance to diffusion) Eqn. [A-1] becomes the Eqn. of wave propagation:

$$\frac{\partial^2 c}{\partial t^2} = V^2 \frac{\partial^2 c}{\partial x^2}$$

Eqn. [A-1] is of the form of the telegraph equation with no leakage:

$$KL \frac{\partial^2 u}{\partial t^2} + (KR + LS) \frac{\partial u}{\partial t} + RSu = \frac{\partial^2 u}{\partial x^2} \quad [A-2a]$$

For $S = 0$,

$$KL \frac{\partial^2 u}{\partial t^2} + KR \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad [A-2b]$$

hence $D = \frac{1}{KR}$ and $V = \frac{1}{KL}$ in the telegraph equation. [A-3]

The source function solution for a finite source of strength $2C_0$ at $x = 0$, $t = 0$ becomes for Eqn. [A-1]:

$$c(t, x) = \frac{e^{-\frac{tV^2}{2D}}}{\frac{2D}{V}} \left[I_0(Y) + \frac{tV^2}{2D} \frac{I_1(Y)}{Y} \right] \quad [A-4]$$

where $Y = \frac{\sqrt{V^2 t^2 - x^2}}{\frac{2D}{V}}$ and I_0, I_1 are the modified Bessel functions of the first kind of orders zero and one.

The semi-infinite constant potential function solution is:

$$\begin{aligned} c(t, x) &= 1 - \int_0^x c(t, x) dx \\ &= 1 - \frac{e^{-\frac{tV^2}{2D}}}{\frac{2D}{V}} \int_0^x \left[I_0(Y) + \frac{tV^2}{2D} \frac{I_1(Y)}{Y} \right] dx \quad [A-5] \end{aligned}$$

where $0 < x \leq vt$, and $C = 0$, where $x > vt$

For $vt \gg x$

$$Y \xrightarrow{t \rightarrow \frac{V^2 t}{2D}} = S$$

the solution becomes:

$$c(t, x) = 1 - \left[\frac{V}{2D} \frac{I_0 S + I_1 S}{e^S} \right] x \quad [A-6]$$

for t large, for $S > 10$, $I_0 S, I_1 S \rightarrow \frac{e^S}{\sqrt{2\pi S}}$

hence

$$c(t,x) = 1 - \frac{x}{\sqrt{\pi Dt}} \quad [A-7]$$

(Compare with solution of classical diffusion equation for t large:

$$c(t,x) = 1 - \operatorname{erf} \frac{x}{2\sqrt{Dt}} \rightarrow 1 - \frac{x}{\sqrt{\pi Dt}}$$

For preparing plots of the solution of $c(t,x)$ for the telegraph equation use of an alternative form to Eqn. [A-5] simplified the calculations, since I_0, I_1 are not tabulated for large values of the argument.

Carson (C-8) pp. 109-113 gives the solution in the form:

$$c(t,x) = e^{-\frac{Vx}{2D} + \frac{V^2x}{2D} \int_{\frac{x}{V}}^t \frac{e^{-\frac{V^2t}{2D}} I_1\left(\frac{V}{2D} \sqrt{V^2t^2 - x^2}\right)}{\sqrt{V^2t^2 - x^2}} dt} \quad [A-8]$$

Let $\tau = vt, \quad a = \frac{V}{2D}$

hence

$$c(t,x) = e^{-ax + a^2x \int_x^\tau \frac{e^{-a\tau} I_1 R}{R} d\tau} \quad [A-9]$$

where $\tau \geq x$ and $R = \sqrt{a^2 \tau^2 - a^2 x^2}$, and $c(t,x) = 0$ for $\tau < x$

Eqn. [A-9] can be solved for values of $a\tau = \frac{V^2t}{2D}$ for $ax = \text{constant}$.

The calculations were performed for $ax = \frac{Vx}{2D} = 3, 1, 0.5$ for appropriate

values of $a\tau$, and values of $c(t,x)$ obtained were plotted against

$\eta = \frac{x}{2\sqrt{Dt}} = \frac{ax}{\sqrt{2a\tau}}$ together with the error integral solution for comparison in Plot A.

2. Summary of classical diffusion theory for concentration dependence of diffusion coefficient. Evaluation of diffusion coefficient when it is a function of concentration only.

Boltzmann's method enables the general diffusion equation for variable diffusion coefficient Eqn. [4] to be reduced for the case where D is a function of concentration only (vide Crank, A-11, for proof). For the one-dimensional case:

$$\begin{aligned} \frac{\partial c}{\partial t} &= \frac{\partial}{\partial x} \left(D(c) \frac{\partial c}{\partial x} \right) & [B-1] \\ &= D(c) \frac{\partial^2 c}{\partial x^2} + \frac{dD(c)}{dc} \left(\frac{\partial c}{\partial x} \right)^2 \end{aligned}$$

Substituting $\xi = \frac{x}{\sqrt{t}}$, hence D independent of ξ then

$$\frac{d^2 c}{d\xi^2} + \frac{\xi}{2D} \frac{dc}{d\xi} + \frac{D'}{D} \left(\frac{dc}{d\xi} \right)^2 = 0, \text{ where } D' = \frac{dD}{dc} \quad [B-2]$$

This expression cannot be integrated but may be evaluated experimentally:

then

$$\begin{aligned} \frac{d}{d\xi} \left(D \frac{dc}{d\xi} \right) &= - \frac{\xi}{2} \frac{dc}{d\xi} \\ D(c) &= - \frac{1}{2} \frac{d\xi}{dc} \int_{c_0}^c \xi dc & [B-3] \end{aligned}$$

and $D(c)$ may be evaluated by graphical integration of right-hand side which is obtained experimentally.

Diffusion equation for concentration dependence

$$D = \left(1 + \lambda \frac{c}{c_0} \right) D_0 \text{ where } \lambda = \text{constant} \quad [B-4]$$

The differential equation is the same as that for diffusion with convection where V_x is replaced by $-D_0 \frac{\lambda}{c_0} \frac{\partial c}{\partial x}$. A plot of the experimental data and type curves for values of λ will yield a solution. The differential equation

$$\frac{\partial c}{\partial t} = D_0 \frac{\partial}{\partial x} \left[\left(1 + \lambda \frac{c}{c_0} \right) \frac{\partial c}{\partial x} \right] = \frac{\partial}{\partial x} \left[D_0 \lambda \frac{c}{c_0} \frac{\partial c}{\partial x} \right] + D_0 \frac{\partial^2 c}{\partial x^2} \quad [B-5]$$

may be compared to the diffusion equation with convection:

$$\frac{\partial c}{\partial t} = - \frac{\partial}{\partial x} (V_x c) + D_0 \frac{\partial^2 c}{\partial x^2} \quad [B-6]$$

so that V_x in [B-6] is equivalent to $-\lambda \frac{D_0}{c_0} \frac{\partial c}{\partial x}$ in [B-5].

An approximate solution of [B-5] by the method of moments is given by Crank (A-11, p. 184)

$$c(x,t) = (\alpha\beta - \frac{3}{2}) \left\{ x\sqrt{\frac{\beta}{D_0 t}} - 1 \right\}^2 + (\alpha\beta - \frac{5}{2}) \left\{ x\sqrt{\frac{\beta}{D_0 t}} - 1 \right\}^3$$

by substituting $\eta = \frac{x^2}{4D_0 t}$

$$c(x,t) = (\alpha\beta - \frac{3}{2}) \left\{ 2\sqrt{\eta\beta} - 1 \right\}^2 + (\alpha\beta - \frac{5}{2}) \left\{ 2\sqrt{\eta\beta} - 1 \right\}^3$$

where β is taken as the root tending to $\frac{1}{12}$ as D tends to D_0 , of the quadratic equation:

$$24\alpha\beta^2 F(1) + \left\{ \alpha - 108 F(1) \right\} \beta + 3/2 = 0 \quad (\text{loc. cit. p. 180})$$

where $F(c/c_0) = 1 + \lambda c/c_0$ hence $F(1) = 1 + \lambda$

and $\alpha = 30 (1 + \lambda/2)$

The solutions for values of $\lambda = 0, 0.25, 0.5, 1, 1.5, 2.5$ has been worked out and is compared in Plot B with the error integral solution, for example:

$$\lambda = 1, \text{ then } \alpha = 45, F(1) = 2, \beta = 0.0691$$

$$c(x,t) = 1.61 \left\{ 0.526\sqrt{\eta} - 1 \right\}^2 + 0.61 \left\{ 0.526\sqrt{\eta} - 1 \right\}^3$$

It is seen that the solution is a close approximation by comparing the solution for $\lambda = 0$ with the exact solution (error integral).

3. Summary of turbulent diffusion theory due to Frenkiel (following the pioneering work of Taylor and Kampe de Fariet).

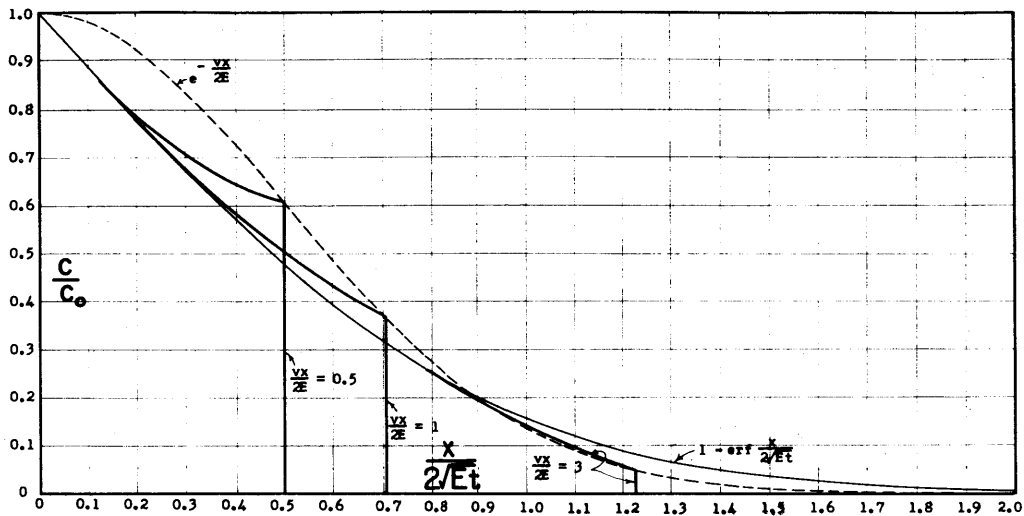
Let U_A denote the mean velocity of a particle A in a homogeneous, isotropic and stationary turbulent field. The Lagrangian correlation coefficient is given by:

$$R_h(h) = \frac{U'_A(t) U'_A(t+h)}{\sqrt{[U'_A(t)]^2 [U'_A(t+h)]^2}} \quad [C-1]$$

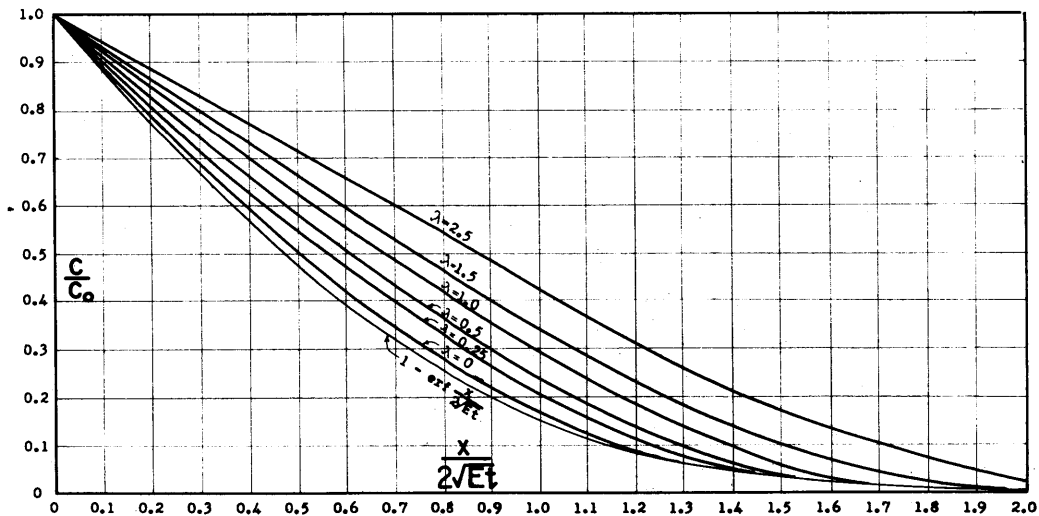
The Lagrangian scale of turbulence is

$$L_h = \int_0^\infty R_h(\alpha) d\alpha \text{ in units of time} \quad [C-2]$$

The fundamental equation of turbulent diffusion is found (first by Kampe de Fariet)



- A. Comparison of solution of classical diffusion equation with solution of Goldstein's turbulent diffusion equation for values of parameter $\frac{vx}{2E} = 0.5, 1, 3$. (Calculated by means of formulation of solution given by Carson, Electrical Circuit Theory and Operational Calculus, 1926, p. 103).



- B. Comparison of classical diffusion equation solution for constant E with solution for concentration dependence of E , where $E = (1 + \lambda c/c_0) E_0$ for values of $\lambda = 0, 0.25, 0.5, 1.0, 1.5, 2.5$. (Calculated by means of approximate method of moments given by Crank, Mathematics of Diffusion, 1956, p. 184).

from a consideration of the variance $\overline{\xi^2}$ (mean square displacement) as a function of the dispersion time t , for one-dimensional diffusion of a finite dosage (source-function):

$$\overline{\xi^2} = 2\overline{V^2} \int_0^t (t - \alpha) R_h(\alpha) d\alpha \quad [C-3]$$

where ξ = the component of displacement in the diffusion direction of a typical particle from its original position in time t .

V = the component of particle velocity in the diffusion direction (Eqn. [C-3] is easier to use in applications than that due to Taylor viz.:

$$\overline{\xi^2} = 2\overline{V^2} \int_0^t d\alpha_2 \int_0^{\alpha_2} R_h(\alpha_1) d\alpha_1 \quad [C-3a]$$

Large diffusion times.

Consider firstly what happens for large diffusion times $t > L_h$.
Eqn. [C-3] becomes:

$$\begin{aligned} \overline{\xi^2} &\approx 2\overline{V^2} L_h t - 2\overline{V^2} \int_0^\infty \alpha R_h(\alpha) d\alpha \\ &\approx 2\overline{V^2} L_h t - \text{constant} \end{aligned} \quad [C-4]$$

for $t \gg L_h$ $\overline{\xi^2} \approx 2\overline{V^2} L_h t$ [C-5]

On comparing with Eqn. [47] of the classical diffusion theory which is identical to [C-5], it is seen that the turbulent diffusion theory yields the same result as the classical diffusion theory for constant diffusion coefficient when $t \gg L_h$, hence for $t \gg L_h$

$$\overline{\xi^2} \approx 2D_{\text{turb}} t$$

where D_{turb} is the characteristic turbulent diffusion coefficient.

Small diffusion times.

For $t \ll L_h$ Eqn. [C-3] becomes:

$$\overline{\xi^2} \approx \left[1 - \frac{1}{6} \frac{t^2}{\lambda_h^2}\right] \overline{V^2} t^2 \quad [C-6]$$

where λ_h = the Lagrangian microscale of turbulence

for $t \ll \lambda_h$

$$\overline{\xi^2} \approx \overline{V^2} t^2 \quad [C-7]$$

(Comparing Eqn. [C-7] with Goldstein's approximate turbulent diffusion theory it is seen that the r.m.s. velocity of migration here is replaced by a finite and constant migration velocity in Goldstein's theory).

General Case.

Rewriting [C-3]:

$$l^2 = 2 \int_0^{\tau} (\tau - \alpha) R_h(\alpha) d\alpha \quad [C-8]$$

where $l = \frac{1}{L_h} \sqrt{\frac{Y^2}{V^2}}$ and $R_h\left(\frac{h}{L_h}\right) = R_h(h)$

Differential Equations for Turbulent Diffusion.

Substitution of the coefficient D in the classical Fickian diffusion equations [Eqns. 1 to 4] in Chap. II by the appropriate turbulence parameter yields the fundamental turbulent diffusion differential equations. Hence Fick's Second Law of diffusion Eqn. [2] is transformed to the corresponding form for turbulent diffusion by replacing

$$\frac{d}{dt} (2Dt) \text{ by } \frac{d}{dt} (\overline{\xi^2})$$

hence replacing D by $\frac{1}{2} \frac{d}{dt} \overline{\xi^2} = n^*(t)$, known as the factor of turbulent diffusion. Thus the diffusion equation [2] Chapter II becomes for a homogeneous, isotropic and stationary turbulent field:

$$\frac{\partial \bar{c}}{\partial t} = n^*(t) \nabla^2 \bar{c} \quad [C-9]$$

where \bar{c} is a function of x, y, z, t

$$\text{and } n^*(t) = \frac{1}{2} \frac{d \overline{\xi^2}}{dt} = \overline{V^2} \int_0^t R_h(\alpha) d\alpha$$

Exact solutions for the differential equations of the statistical theory

of turbulent diffusion are of the same form as the corresponding classical diffusion solution (for constant D) except that the quantity $2Dt$ is replaced by the variance $\overline{\xi^2}$ which is given by Eqn. [C-3]. Hence, for example, the three-dimensional source function solution for classical (molecular or Brownian) diffusion is:

Eqn. [45]

$$c = \frac{M}{(4\pi Dt)^{3/2}} e^{-\frac{r^2}{4Dt}} \quad [C-11]$$

and for turbulent diffusion the corresponding solution is:

$$c = \frac{M}{(2\pi \overline{\xi^2})^{3/2}} e^{-\frac{r^2}{2\overline{\xi^2}}} \quad [C-12]$$

where $\overline{\xi^2}$ is given by Eqn. [C-3].

The two expressions above are identical when

$$2Dt = \overline{\xi^2} \quad [C-13]$$

which is only so when $t \gg L_h$ as stated before.

In a turbulent fluid the diffusion is a result of the combined action of molecular and turbulent activity hence $D_{\text{turb}} = D_{\text{mol}} + E$, where E is the eddy diffusivity. Assuming that the Schmidt number is unity, one may write $D_{\text{mol}} = \nu$, $E = \epsilon$ for usual turbulence levels $\epsilon \gg \nu$ hence one may write:

$$(\text{for } t \gg L_h) \quad \overline{\xi^2} = 2Et \quad [C-14]$$

which reduces the turbulent diffusion equation to the classical diffusion equation with constant diffusion coefficient.

C. Parameters for determining eddy diffusivity (functional relationship with turbulence level).

1. Dynamics of turbulence generating mechanism, measurement of power input.

In order to determine the relationship between the turbulence characteristics and the eddy diffusivity, a parameter for measuring the turbulence was required. The rate of doing work on the fluid is such a parameter and is readily determinable.

The truss with attached screens is supported at four points in such a fash-

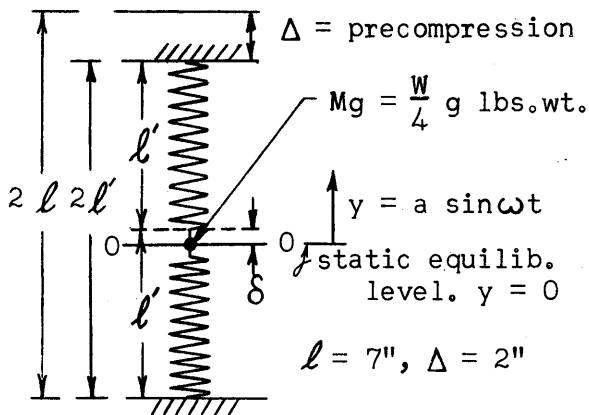
ion that it is free to perform oscillations under the action of an applied harmonic force of variable amplitude and frequency.

Consider the spring mass system in the accompanying figure, with M representing one of the four reaction points of the truss.

A pair of compression springs at each reaction point stores the kinetic energy of the moving point mass M into potential energy with each excursion from the static equilibrium level OO and restores it to the point mass on the return towards OO. In order that the springs will always be under compression, they were precompressed a total distance $\Delta > 2(a + \delta)$.

The external work stored per 1/4 cycle by the springs becomes zero as the frequency of oscillation approaches the natural frequency, since at resonance the potential energy stored + kinetic energy = $Mga = \text{constant}$ and external force is zero. Let each spring constant be equal to K, total precompression = Δ or per

spring = $\frac{\Delta}{2}$, thus precompressed force at each support = $K \frac{\Delta}{2}$. Equivalent spring constant per support for precompression = $\frac{K\Delta}{2} \frac{1}{\Delta} = \frac{K}{2}$. (On the other hand, for displacement at the midpoint, the equivalent spring constant is $2K$, hence static deflection $\delta = \frac{Mg}{2K}$).



Determination of spring and inertia forces.

In order to determine the magnitude of $4R$, the spring force minus the inertia force of the moving mass, so that this may be subtracted from F_T , the total force measured, so as to yield the hydrodynamic force, it

was necessary to calculate or experimentally determine the natural frequency of oscillation σ_{nat} of the spring - mass system. The truss could not be allowed to reach resonance in the absence of the damping effect of the water, so that the experimental determination of σ_{nat} had to be indirectly made. The maximum connecting rod force was namely obtained for various frequencies of oscillation of the truss and attached screens in the absence of water in the flume. The natural frequency could be obtained from Eq. 62 or graphically (Plot 1).

The equation of motion for free oscillations without damping of the spring mass system at each support is:

$$M\ddot{x} + 2Kx = 0$$

and the natural frequency given by:

$$\sigma_{\text{nat}} = \frac{p}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{2K}{M}} \quad [61]$$

where p = angular velocity at resonance.

For forced oscillations without damping the actuating force per support is given by:

$$R = (2K - M\omega^2) a \sin\omega t$$

where ω = angular velocity; a = amplitude of forced vibration.

Substituting from Eq. [61]:

$$M = \frac{2K}{p^2},$$

$$R = 2Ka \left(1 - \frac{\omega^2}{p^2}\right) \sin\omega t \quad [62]$$

Determination of dissipative energy input for forced oscillation with screens submerged in water.

The total instantaneous hydrodynamic force due to the damping effect of the water is equal to the virtual mass force plus the drag force and may be written as:

$$F = 1/2\rho C_D^i (\text{total solid screen area}) a^2\omega^2 |\cos\omega t| \cos\omega t - M' a^2 \sin\omega t$$

$$D = 127 C_D^i a^2\omega^2 |\cos\omega t| \cos\omega t - 1.29 a\omega^2 \sin\omega t \quad [63]$$

where C_D^i = drag coefficient based on total solid screen area and instantaneous screen velocity

M' = virtual mass of screens assumed equal to mass of water displaced by screens

The maximum value of F_D occurs as $\cos\omega t = 1$, because the coefficient of the second term in Eq. [63] is small compared to the first term, hence,

$$F_{D_{\text{max}}} = 127 C_D^i a^2\omega^2 \quad [64]$$

The instantaneous power input, equal to the dissipative force times the velocity at the point of application, is given by:

$$P = 550 \text{ (H.P.)} = F_D a \omega \cos\omega t \quad [65]$$

and the mean power input in ft. lbs. per second by:

$$\bar{P} = \frac{1}{2\pi} \int_0^{2\pi} P d(\omega t) = \frac{1}{2\pi} \int_0^{2\pi} [1.29 a\omega^2 \sin\omega t + 127 C_D^i a^2\omega^2 \cos^2\omega t] d\omega t$$

$$= \frac{254}{\pi} C_D^i a^3\omega^3 \int_0^{\pi/2} \cos^3 u du = 54 C_D^i a^3\omega^3 \quad \text{ft.-lbs./sec.} \quad [66]$$

where C_D = drag coefficient based on total solid screen area and mean screen velocity and

in which the first term vanishes as integrated from 0 to 2π , hence power proportional to $(a\omega)^3$ (See Plots 3 and 4).

2. Relationships between power-input, average eddy viscosity, turbulence intensity.

The eddy viscosity in the y-direction (vertical) may be expressed by

$$E_y = \bar{v} l_y$$

where \bar{v} denotes $\sqrt{v'^2}$ and l_y is a vertical mixing length. Since \bar{v} , l_y are statistical properties of the turbulence which can not be simply evaluated, the assumption is made that l_y is proportional to the maximum travel distance of the screens, $2a$, and v is proportional to the r.m.s. velocity of the screens. Hence one may define a vertical "mechanical eddy viscosity" as:

$$\begin{aligned} (E_y)_{\text{mech}} &= \frac{\omega a}{\sqrt{2}} \cdot 2a \\ &= 2\sqrt{2}\pi\sigma a^2 \end{aligned} \quad [67]$$

For the eddy viscosity in the x-direction (longitudinal), correspondingly expressed by $E_x = u l_x$, it is assumed that u is proportional to v and l_x is proportional to a characteristic screen dimension such as the strand width, b , or possibly the mesh size.

Hence one may similarly define a horizontal "mechanical eddy viscosity" as

$$\begin{aligned} (E_x)_{\text{mech}} &= \frac{\omega a}{\sqrt{2}} \cdot b = \sqrt{2}\pi b a \sigma \\ &= \text{const} (a\sigma) \end{aligned} \quad [68]$$

From Eqn. [66] the mean energy dissipation per unit volume:

$$\frac{\bar{P}}{V} = \bar{W} = \text{const.} (a\omega)^3 \quad [66a]$$

hence one obtains:

$$\begin{aligned} (E_x)_{\text{mech}} &= \text{const.} (\bar{W})^{1/3} \\ &= \frac{\sqrt{2}}{(550 \times 16.3 C_D)^{1/3}} (\text{H.P.})^{1/3} \end{aligned} \quad [69]$$

Eqn. [69] quantitatively expresses the well-known fact that work done on

a fluid at rest can only be ultimately dissipated by viscous shear which requires small scale velocity gradients at the fluid boundary and throughout the fluid. These gradients are created by the turbulent fluctuations set up as a result of the condition that the total circulation in irrotational flow is a constant; equal to zero in the case of a fluid at rest.

Expressions for work input per unit volume (rate of energy dissipation) may be formulated, for instance, in terms of mean velocity gradients (Ignacio, H-5 and Camp, H-6):

$$\bar{w} = 15 \mu \left(\frac{\partial u}{\partial x} \right)^2 \quad [70a]$$

or in terms of turbulent velocity fluctuations (Goldstein, H-2):

$$\bar{w} = 15 \mu \frac{\overline{u'^2}}{\lambda^2} \quad [70b]$$

where λ = a characteristic scale of turbulence.

3. Modifications of diffusion coefficients by gravity effects due to density gradients.

The above expressions for the eddy diffusivity are not valid where gravitational convection due to non-uniformity in the fluid density aids the diffusion process.

4. Parameters for diffusion coefficients modified by gravity effects.

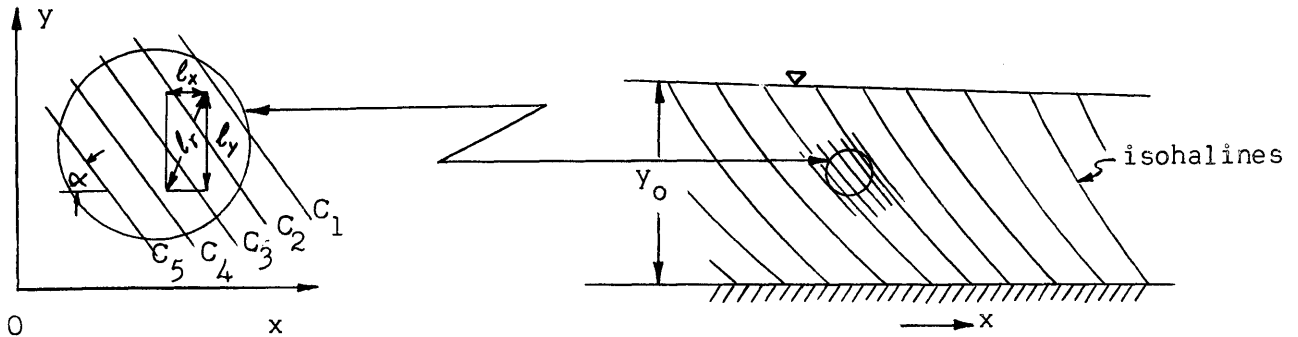
In partially mixed estuaries, even if there were no tidal currents, gravity forces would cause concentration gradients with depth, since the unbalanced forces due to the variation in density from fresh water to ocean water and the landwards diffusion process of salt below mid-depth and retards it above mid-depth. The tidal currents tend to further increase the depth-dependence of the diffusion and convection process and hence causes increased vertical concentration gradients. In general, for a partially mixed estuary it may be stated that the isohalines (lines of equal salt concentration) will be neither horizontal (completely stratified) nor vertical (well-mixed). It remains to be here investigated whether a one-dimensional approach may be justified in a diffusion process where the isohalines have a positive slope $\frac{\partial y}{\partial x}$ where x is measured in the oceanward direction and y is measured upwards.

By definition the turbulent eddy diffusivity is given by:

$$u'c' = E \frac{\partial c}{\partial x}$$

where u', c' denotes here r.m.s. of turbulent fluctuations of velocity and concentration.

$$E = \frac{u'c'}{\frac{\partial c}{\partial x}}$$



Definition sketch for longitudinal cross diffusion coefficient when: $\frac{\partial c}{\partial y} \neq 0$, $\frac{\partial c}{\partial x} \neq 0$.

One may also consider the r.m.s. turbulent concentration fluctuation as caused by a mean displacement l of the mean concentration along a concentration gradient $\frac{\partial c}{\partial x}$.

$$c' = l_x \frac{\partial c}{\partial x},$$

thus

$$E = u' l_x = E_x$$

Consider now the case where one has non-isotropic turbulence, as well as the conditions existing in a "partially mixed" estuary. Let the angle that the isohalines make with the horizontal be α . Let the characteristic vertical eddy diffusivity coefficient be $E_y = v' l_y$ and the characteristic horizontal eddy diffusivity be $E_x = u' l_x$, where E_x , E_y both may be functions of x , y and t .

Let

$$\frac{l_y}{l_x} = \tan \beta$$

and

$$l_r = (l_x^2 + l_y^2)^{1/2}$$

Consider quasi-steady state, $\frac{\partial c}{\partial t} = 0$,

$$\begin{aligned} c' &= l_r \frac{\partial c}{\partial r} = l_r \left(\frac{\partial c}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial c}{\partial y} \frac{\partial y}{\partial r} \right) \\ &= l_x \frac{\partial c}{\partial x} + l_y \frac{\partial c}{\partial y} \end{aligned}$$

In the experimental case of oscillating screens producing the turbulence,

assume

$$\frac{l_x}{l_y} = \frac{b}{2a}$$

where b =strand width, a = amplitude.

Denote diffusion coefficient, modified due to sloping isohalines, gross diffusivity by E' , hence:

$$E'_x = \frac{c|u|}{\frac{\partial c}{\partial x}}$$

and

$$\frac{E'_x}{E_x} = 1 + \frac{2a}{b} \cot\beta$$

IV EXPERIMENTAL EQUIPMENT

A. Practical considerations leading to the choice of boundary configuration.

The availability of a ready mathematical solution (the error integral) for diffusion in a semi-infinite one-dimensional medium, the boundary of which is maintained at constant concentration potential and the extensive occurrence of this boundary condition in the literature on diffusion, as well as its adaptability to available apparatus, led to its choice for determining diffusion coefficients. (Other boundary configurations have been considered, one of which is the two-dimensional diffusion from a source of constant strength of diffusion flow.) Choice of the one-dimensional semi-infinite case, however, would enable anticipated future studies of estuary diffusion problems to be performed by means of simple adaptations. The infinite length requirements of the mathematical model are satisfied by experimental set-up dimensions, as long as the concentrations at the far end have not attained a measurable value, after which time the boundary would reflect back the virtual concentration growth beyond it. By the method of reflection, records could also be analyzed where finite concentrations were allowed to build up at the far end, had a 32 foot long flume not been available.

B. Apparatus. Choice of turbulence producing element.

The relative merits of various devices for producing uniform turbulence were considered. It was desired to generate a homogeneous, approximately isotropic turbulence, which is best achieved in fluid motion by some type of screen or grid. Thus a screen element was sought which was lightweight, yet sturdy, displaced only a small fraction of the water, did not obstruct the flow in its plane by more than fifty percent solidity, was resistant to corrosion, and could be made to produce desired levels of turbulence by oscillating it with small amplitude about the mean position in the fluid at rest. The power input may be measured readily by means of a strain gage load cell on the actuating arm. Turbulence of the type produced by moving blades, paddles or stirring devices was not desired since any vortex tubes formed would cause increased vertical mixing lengths counteracting density effects and would not allow partially mixed or stratified conditions to be obtained in estuary studies.

Mechanical considerations led to the choice of a truss-type girder for supporting the turbulence producing screens so that the entire unit could be made to oscillate with desired frequency and amplitude by applying a simple harmonic force at a central point. A vertical simple harmonic motion was selected as most practicable.

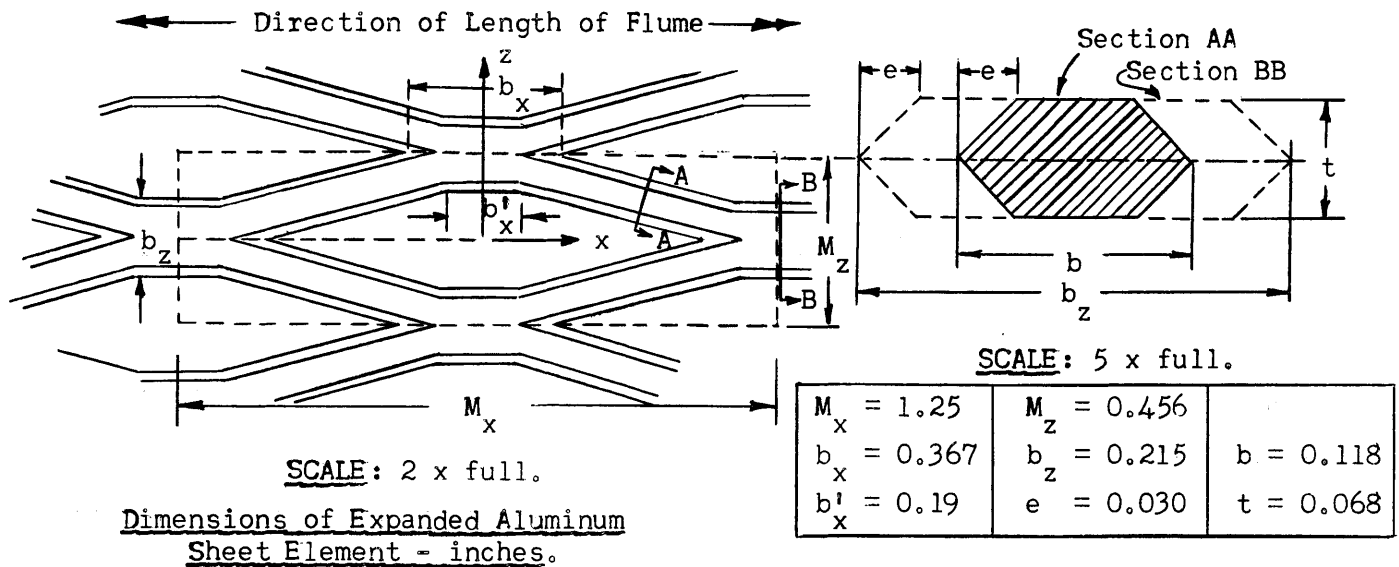
For the turbulence producing screens the following possibilities were investigated: (i) woven wire cloth, (ii) perforated plate, (iii) expanded sheet metal. Materials considered were, for (i) brass or copper; for (ii) bronze, monel, stainless steel, rust-proofed steel or aluminum; for (iii) steel or aluminum. Cost considerations ruled out perforated plate in any metal but steel; however, adequate corrosion protection of the steel in salt water could not be obtained.

(i) A sample of woven wire cloth was obtained but this was rejected due to its

lack of mechanical rigidity and its low turbulence producing ability, ascertained by preliminary tests.

(ii) Samples of square-opening perforated plate were obtained in steel in 1/2", 3/2" and 1" side-of-opening sizes with open to closed area ratio about 0.65. Only the 1/2" size was considered a possibility. Tests made by observing the fall velocity of a sample piece in a large tank of water and studies of its effect on dye particles, indicated that it would generate a suitable degree of turbulence and produce measurable forces per unit area. It was given two coats of aluminum paint but two days' immersion in salt water showed peeling of the paint and rusting. Anticipating further trouble with steel, this material was ruled out, and since the next in cost, aluminum, was two or three times as expensive, perforated plate was impractical. The cost of aluminum expanded metal was substantially less than aluminum perforated plate of similar characteristics, and after preliminary trials with a sample sheet, six sheets, altogether eight feet by four feet nominal size, were obtained from which the stack of screens was assembled: 30 ft. overall length by 16" wide, 6 screens high, at 1" spacing.

The dimensions of a screen element was as follows, as determined by travelling microscope, engineer's scale and micrometer.



Turbulence Generating Mechanism. The accompanying figures show the general appearance and layout of the apparatus.

The screens were supported from a truss made up from two Bethlehem steel floor joists, rigidly torsion-braced and supported on four posts, located at the quarter points of the truss length, between a pair of compression springs at each post. Guides for the motion of the support points are formed by pillow blocks housing ball bushings, running on hardened polished stainless steel spindles (Fig. 3). Each spindle is threaded into a blind flange capping the supporting column, en-

APPARATUS FOR
TURBULENT DIFFUSION
STUDIES

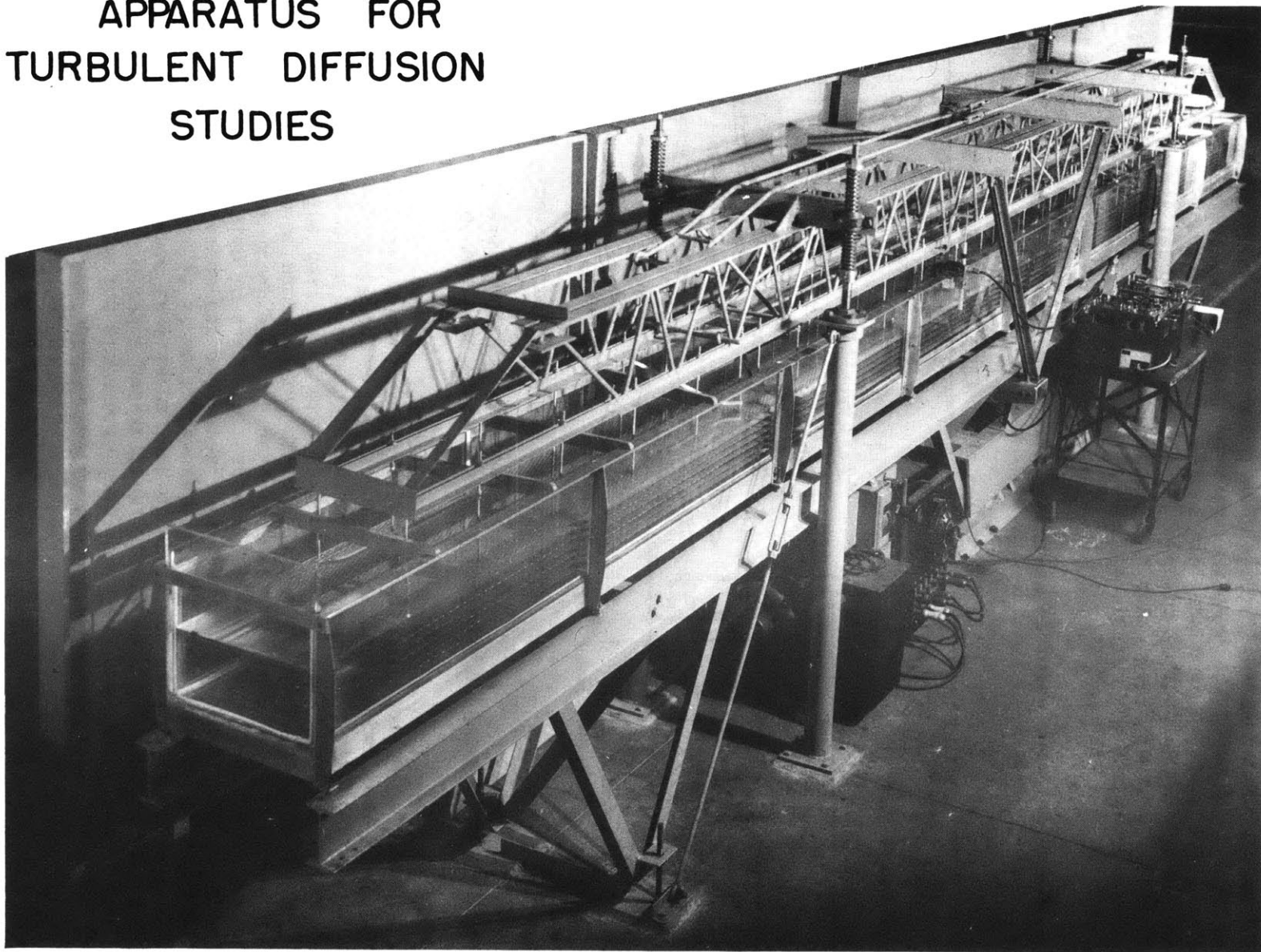


FIG. I

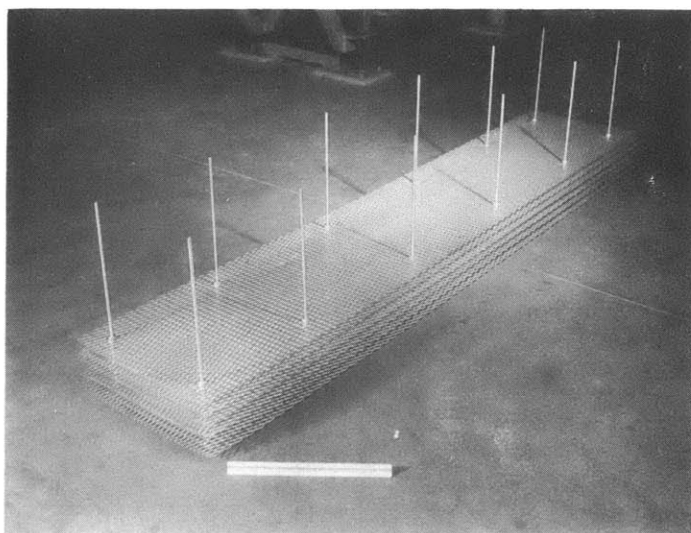


Fig. 2. 8" - 4" x 16" assembled section of turbulence producing screens, consisting of 6 Expanded Aluminum sheets at 1" spacing.

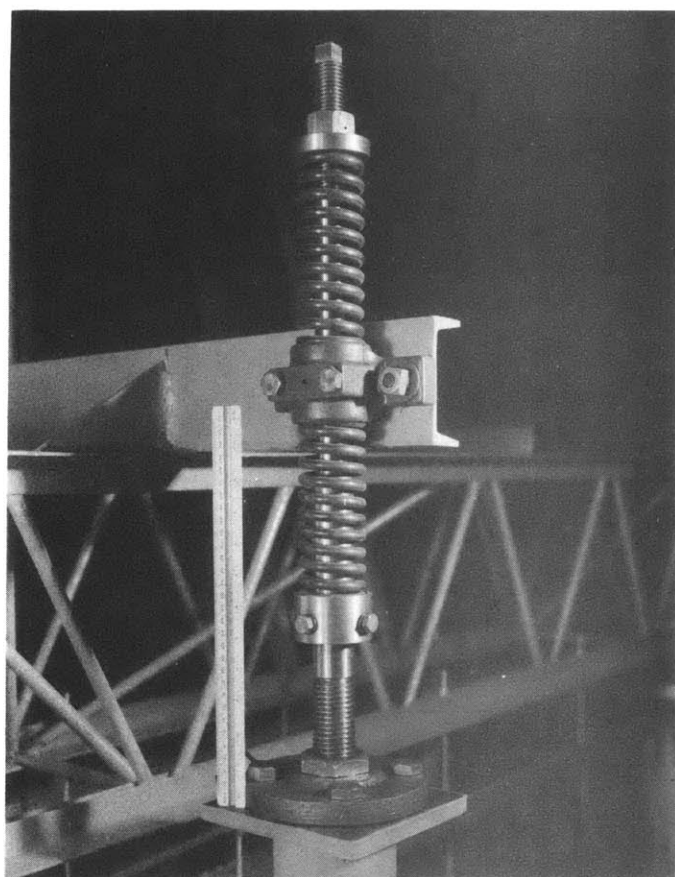


Fig. 3. View of Mode of Support of Truss showing compression springs, Ball bushing housing and spindle.

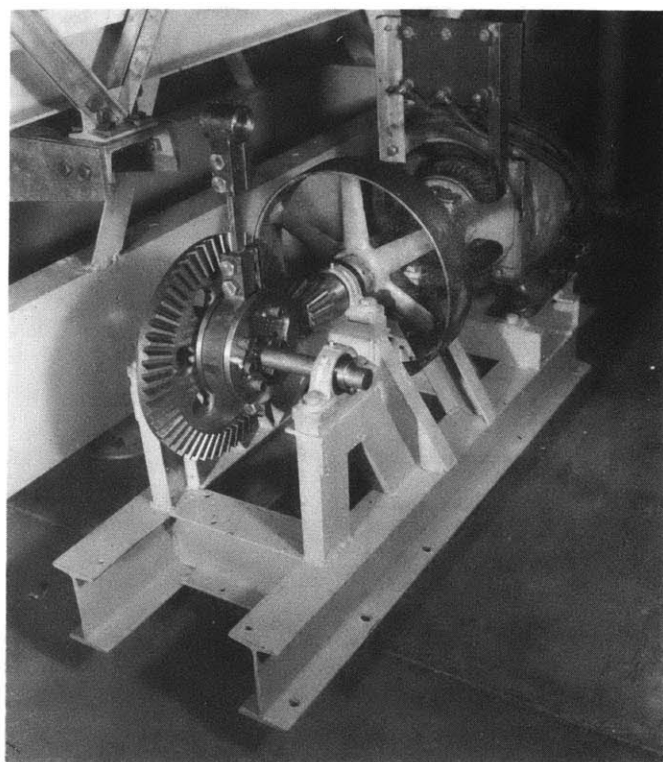


Fig. 4. Drive Unit before assembly, showing eccentric and connecting rod. (Strain gages not yet mounted when picture taken).

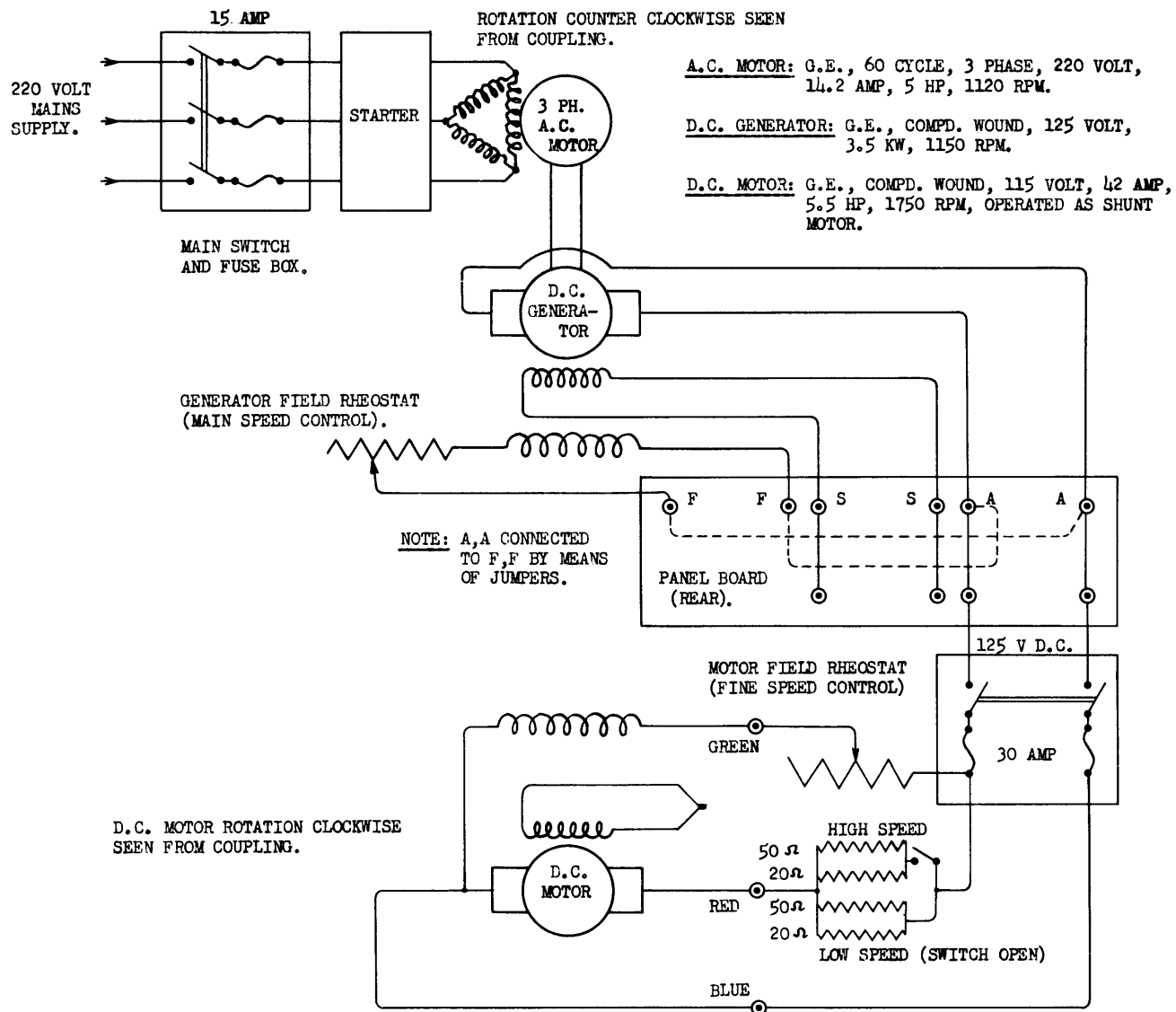


FIG. 5. CIRCUIT DIAGRAM FOR A.C. - D.C. WARD - LEONARD SPEED CONTROL SYSTEM, FOR DRIVING TURBULENCE GENERATOR MECHANISM.

abling adjustments for leveling the support points. The loading bars consists of four members attached at their upper ends to approximately the third points of the center span and at their lower ends to a transverse crossbeam (Fig. 1). A vertical component of force at the center of the cross beam will displace the entire truss with attached screens as a unit vertically an equal distance at each of the four supporting points. The power supply consists of a 3.5 K.W. Ward-Leonard A.C.- D.C. speed control unit (Fig. 5) and the drive unit consists of a 5.5 H.P. variable speed D.C. motor, coupled to a pinion and bevel gear (4:1) reduction, with an adjustable stroke eccentric cam mounted directly on the bevel gear. The driving components are assembled in a bed frame bolted to the floor, with the cam shaft directly below the load point of the cross beam, attached to the truss. A connecting rod which constitutes also the force gage, transfers the eccentric rotation of the cam into what may be considered, for all practical purposes, a simple harmonic motion of the cross beam. Friction losses were made negligible by means of needle bearings inside the eccentric follower (big end) and self-aligning ball bearings at the load transfer point (small end) and in the shaft supports of the drive unit. The friction in the guides was made negligible by the use of ball bushings.

The force is measured directly by means of four strain gages, mounted on the connecting rod link which was made of a sufficiently small cross section so that measurable strains would be obtained. The force gage is fully described in Section C.

The laboratory flume that was available for the diffusion experiments is a 32 feet long by 16 1/2" inside width, by 13" inside height, lucite-wall channel mounted on a steel frame which may be set horizontal or tilted to slopes of several degrees either way. There is no physical contact between the turbulence generator and the flume, except through the water in the flume and through the floor. Thus vibrations of the flume are minimized and the motion of the water is solely due to the turbulence produced by the oscillating screens. Fig. 1 shown an overall view of the completed apparatus, while some of the components are shown in close-up views, Figs. 2, 3, 4.

C. Instrumentation.

Force gage. The force gage consisted of the link of the connecting rod, coupling the eccentric cam and the cross-beam. Four strain-gages were cemented onto the tension-compression link in a "Double-Poisson" arrangement to form the elements of a Wheatstone Bridge (Fig. 6). The strain-gages were of type A-7, 120 ohm, gage factor $1.89 \pm 2\%$, SR 4, manufactured by Baldwin-Lima-Hamilton.

Double-Poisson mounting for axial force gage. The Double-Poisson arrangement had the following desirable features, making it well suited for axial force measurements:

- 1) All active components are subjected to the same atmospheric conditions and are electrically symmetrical and balanced. The gage is thus fully temperature compensated.
- 2) Signal amplification of 2.6 times that of a single strain gage is obtained

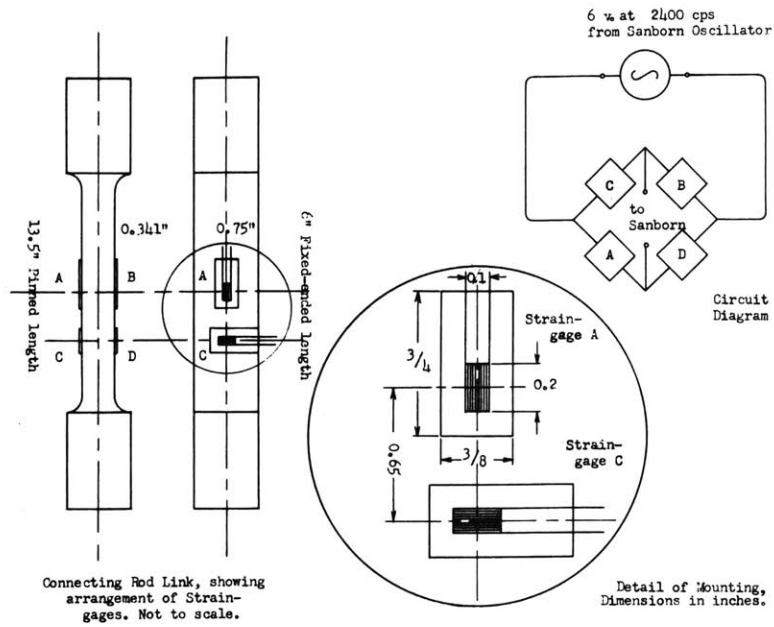


Fig. 6. Double-Poisson Strain-gage mounting for Force Gage.

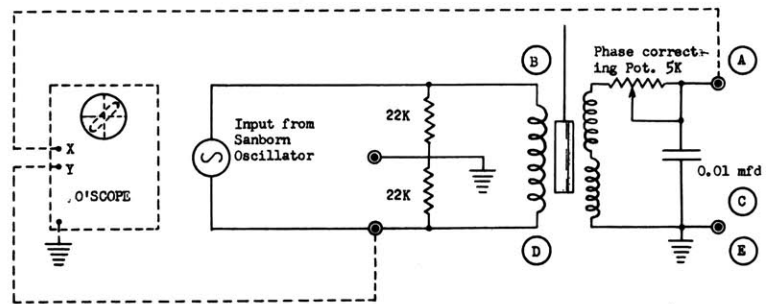


Fig. 7. Phase Correcting Network for Displacement Gage.

(2 x 1.3 using Poisson's ratio for steel = 0.3).

- 3) Insensitivity of the force gage to bending moments (secondary), since tension and compression strains due to moments are cancelled out electrically, the strain-sensitive elements being on diametrically opposed bridge arms.
- 4) Insensitivity to secondary transverse shear since strain gages are located at or close to the center of the link, where shear due to transverse thrust and end fixity is equal to zero.

The output from the force gage was recorded on the lower channel of the "Twin-Viso" Sanborn Recorder on full-bridge after balancing and zero adjustment.

Calibration of the force gage was made by means of a loading beam and 40 lb. weights for tension and by means of a compression test stand, incorporating a hydraulic jack and platform scale for compression. An electrical calibration mark was recorded at the time of calibration on each of the attenuation settings used (x5, x2, x1). Before each series of runs, where force measurements were made, the electrical calibration was recorded for x1 and compared with the original and, if necessary, the gain control could be adjusted to bring them into agreement, thus ensuring that the calibration curve would be valid. This was never done, however, as the calibration was in all cases found to be in excellent agreement.

Displacement gage. The displacement gage consists of a Linear-Variable Differential Transformer of total linear-output travel of four inches, so that it can record an oscillation with maximum amplitude of two inches about the mean. The unit is a Linearsyn Model 7S2 \pm 2", manufactured by "Control Components", Brookline (now incorporated with Sanborn Co.).

A phase-correcting network is used in conjunction with this transformer unit so as to ensure a linear calibration curve with the Sanborn Recorder used (Fig. 7).

The upper channel on the "Twin-Viso" Sanborn Recorder was used for recording the output from the displacement gage. After phase correction was made by means of an oscilloscope, the attenuation was set on X200 and the gain so adjusted as to keep the record on scale. The ordinary nulling procedures could not be achieved but the capacitance and resistance balance controls were set to a satisfactory position. The calibration obtained during Series 2B was plotted and was found to be linear.

The displacement trace was used only as a reference so that the cycle may be correctly determined as starting from mean position to mean position.

For Series 1, 2A, 2B, 3A, 3B, 4, 5, 6, 7 the displacement gage was set up immediately below the N.E. load point.

The magnitude of the amplitude and stroke was obtained by means of a vernier caliper at the N.E. load point for amplitudes exceeding 0.5 inch and by means of an Ames indicator dial gage for amplitudes less than 0.5 inch, for the static,

maximum displacement condition. Readings were taken to the nearest thousandth of an inch.

Recording amplifier. A two-channel "Twin-Viso" Sanborn Recorder, Model 60-1300, incorporating an independent timer, was used for recording the force and displacement gage signals.

Paper speed was selected by inserting the appropriate set of driving gears for high speed of 100 mm per sec., low speed of 10 mm per sec.

The actual high paper speed was found to vary between 80 and 100 mm per sec. depending to some extent on the amount of paper on the roll. Every run analyzed was done so based on the instantaneous paper speed as determined from the spacing of two consecutive one-second timing intervals, opposite to the portion of the run analyzed.

The timer was checked as giving 30 pips during a 30 second interval, when the mean paper speed was about 90 mm/sec. No effort was made to adjust the paper speed to 100 mm/sec, as no need was seen for this.

Excitation of the Sanborn amplifier is 2,400 c.p.s. at 6 v. The frequency response is 100 per cent at up to 100 c.p.s.

Conductivity probe. Six conductivity probes were made up for this study from one ounce metal hypodermic syringes fitted with No. 19 needles, 6" in length specially made (All above components supplied by E. F. Mahady Co.).

The hypodermic needle formed the one electrode, while a 0.01 inch platinum wire coated with Neoprene cement was threaded through the needle to form the other electrode, and connections to shielded lead-in cables were made. The syringe body was then filled with Epoxy Resin ("Scotchcast" Resin No. 2, Minnesota Mining and Manufacturing Co.) and the resin forced by means of the plunger through the annular space between inner and outer electrode.

After the setting of the resin the tip of the probe was ground to a bevel of slope 5 to 1 approx.

The tips of the probes were platinized commercially for a length of about 1/2" after which the entire body and needle of the probe were given three coats of Neoprene cement, except for the platinized tip.

The bevelled surface was coated with platinum black to minimize conductivity changes and hence calibration drift due to polarization.

Recording of diffusion tests. A single-channel Sanborn Industrial Recorder, Model 127 was used for recording the signals from the conductivity probes.

Paper speed was 2.5 mm per sec. for all diffusion tests. A higher speed of 25 mm per sec. could also be obtained which was sometimes used to observe turbulent concentration fluctuations which were very well picked up by the conductivity probes.

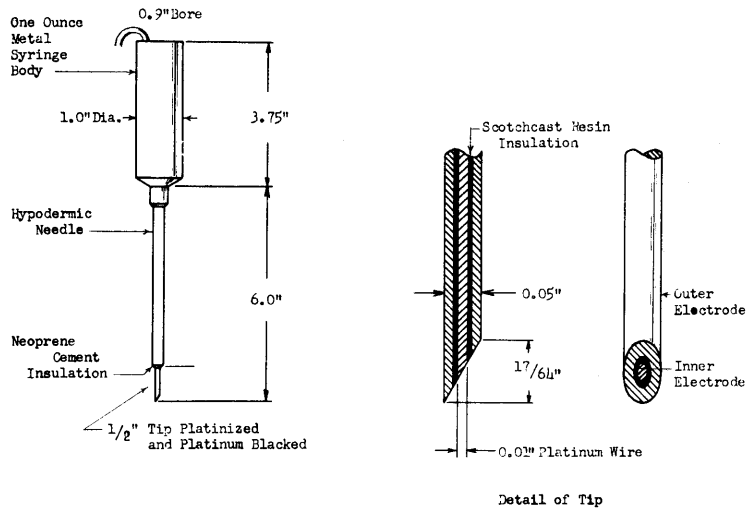


Fig. 8. Conductivity Probe for Measuring Salinity

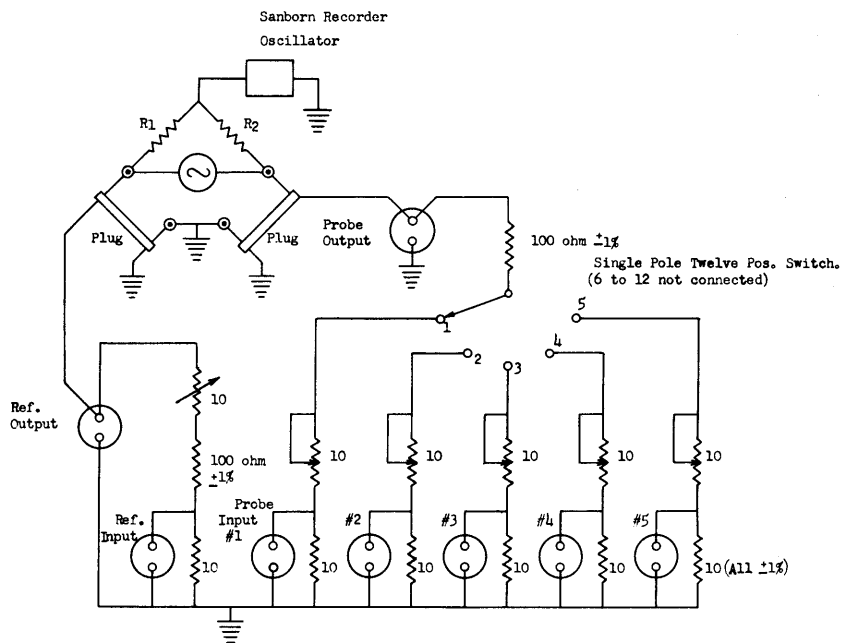


Fig. 9. Circuit Diagram for Conductivity Probe Bridge

Sensitivity of the Sanborn amplifier is 10 ma per cm deflection of stylus.

A twelve position switch was used to switch from probe to probe during a run (A General Radio switch was first used for Runs 10 to 26 and was thereafter replaced with a Leeds and Northrup enclosed switch of better design). The contact resistance had to be less than 1/100 ohm for reliable results.

The measuring and reference probe circuit is shown in Fig. 9. The tip of the reference probe was immersed in a 1.65% salt solution in a 250 ml. container, separate from the diffusion flume.

V EXPERIMENTAL PROCEDURE

A. Determination of power-input as a function of frequency and amplitude.

Objective.

The measurement of the rate of energy dissipation as a function of the independent variables Ω and a constituted the first phase of the experimental program. The rate of dissipating energy, or doing work on the fluid is a measure of the turbulence produced, so that a convenient parameter is hereby obtained, in terms of which the variation in diffusion coefficient with turbulence level may be expressed.

Preparation for measuring power input.

In order to determine the average power input, either the torque and the angular velocity have to be measured or the vertical component of force and the velocity of its point of application. The apparatus was designed for the latter procedure to be followed.

The vertical displacement versus time for the oscillating screens was measured by means of a differential transformer gage which produces an output voltage linearly proportional to the displacement, described in Chapter IV. The displacement gage was mounted to the edge of the flume wall directly below the North-East loading bar attachment point. The movable core of the gage was attached to the lower chord of the truss, so that it was centered with respect to the transformer coils in the body of the gage when the truss was at its mean position. For calibration purposes, before commencing with dynamic tests, an Ames dial gauge was mounted immediately beside the displacement gage. The deflections on the Ames dial were read for one-eighth increments of a revolution of the eccentric and the output signal of the displacement gage recorded. The resulting calibration curve was found to be linear, and being linear it was not necessary to use a calibration curve for the data reduction. Once a desired stroke had been set, a static measurement of the amplitude sufficed.

Since the ratio of the amplitude (1" max.) divided by the connecting rod length (13 1/2") was small, the motion was a close approximation to a simple harmonic motion and the displacement trace practically sinusoidal. The displacement trace

obtained during a run was used as a reference for determining the frequency with the aid of one second timing marks and for determining the phase relationship of the force trace.

During one series, the displacement gage was mounted at different locations along the length of the truss on both sides and the dynamic amplitude as a function of frequency obtained for each location.

It was found that the dynamic behavior, otherwise than the static, tended toward a practically uniform amplitude distribution along the length of the truss. The uniformity was good at one quarter the natural frequency and improved with higher frequencies.

It had been found before commencement of any dynamic tests that the static bending deflection of the truss produced a midspan amplitude (spring deflection plus flexural deflection at the center) which was 28 per cent larger than the amplitude near the end of the cantilevered portion. Tie rods and turnbuckle were thereupon installed to increase the moment of inertia over the central span of the truss and sufficient prestress was applied so as to prevent stress reversal in the tie rods under maximum static load conditions (turnover of cam with a 1" eccentricity). The flexural moment of inertia was effectively doubled over the center span after which improvement the static deflections were again measured and the central amplitude found to be 17 per cent larger than the end amplitude, being a reduction of 40 per cent in the differential deflection. It was decided that further stiffening would be inadvisable, as it would result in cramped working space below the truss, would involve too much added weight, which would further decrease the natural frequency.

The natural frequency for undamped vibration (no water in flume) was experimentally determined by extrapolation (as it was unsafe to exceed 0.75 of the natural frequency in undamped operation due to secondary vibrations) to be 6 c.p.s. before the tie rods were installed and 5.84 c.p.s. after installation.

Power input measuring tests.

Power input as a function of amplitude and frequency was determined from a series of runs made at amplitudes of 0.432", 0.625" and 0.238" over a frequency range which depended on the amplitude, as shown in Fig. 2 ranging from 2.25 to 5.25 c.p.s. Force and displacement traces were recorded for about 10 cycles at each frequency at high recorder speeds (85 to 100 mm/sec.).

The force recorded by the force gage (connecting rod) consisted of the instantaneous value of the total hydrodynamic drag force, plus the total spring force, minus the inertia force of the entire moving mass, minus the virtual mass force. By calculation of the spring and inertia forces, the drag force could be determined. The virtual mass force was negligible.

The velocity of the truss motion times the drag force was calculated and averaged over a cycle to yield the power dissipated in foot lbs./sec.

B. Time-dependent diffusion tests.

Objective.

The unsteady type diffusion tests were performed as Phase II of the experimental research program with the objective of verifying the validity of applying classical diffusion theory to turbulent diffusion and to determine the effects of density difference (between diffusant and receiving medium) on the diffusion process.

Nature of diffusion tests.

The flume could be divided lengthwise by a central barrier which was removed at the instant the diffusion process was to commence. All of the time-dependent diffusion tests were run with the diffusant solution initially contained at uniform concentration in the left half of the flume (Fig. 1). An equal volume of receiving liquid was contained in the right half of the flume in which all concentration measurements were made. The following types of time-dependent diffusion tests have been carried out.

(a) For the case where a density differential is present between diffusant and receiving medium,

1. Salt water (sp. gr. 1.01, 1.65% by weight of salt) diffusing into fresh water (sp. gr. 0.998).
2. Salt water (sp. gr. 1.021, 3.18% by weight of salt) diffusing into fresh water (sp. gr. 0.998).

(b) For the case of zero density differential between diffusant and receiving medium,

3. Salt water (sp. gr. 1.01) diffusing into sugar water (sp. gr. 1.01), 1.65% by weight of salt and 3.00% by weight of sugar.
4. Dyed fresh water diffusing into clear fresh water.
5. Salt water, containing small buoyant polystyrene tracer spheres, diffusing into salt water, the salt water being the same throughout and of specific gravity equal to that of the tracer spheres (sp. gr. 1.05).

The tests outlined above were run at various combinations of screen frequency $\sigma = \frac{\omega}{2\pi}$ and amplitude, a , corresponding to various values of the eddy viscosity defined as the mixing length b , times the r.m.s. value of the screen velocity:

$$E_x = b\sqrt{v^2}$$

where

$$v = a\omega\cos\left(\frac{\omega t}{2\pi}\right)$$

$$E_x = \frac{ab\omega}{\sqrt{2}} = \sqrt{2}\pi ab\sigma = k(a\sigma)$$

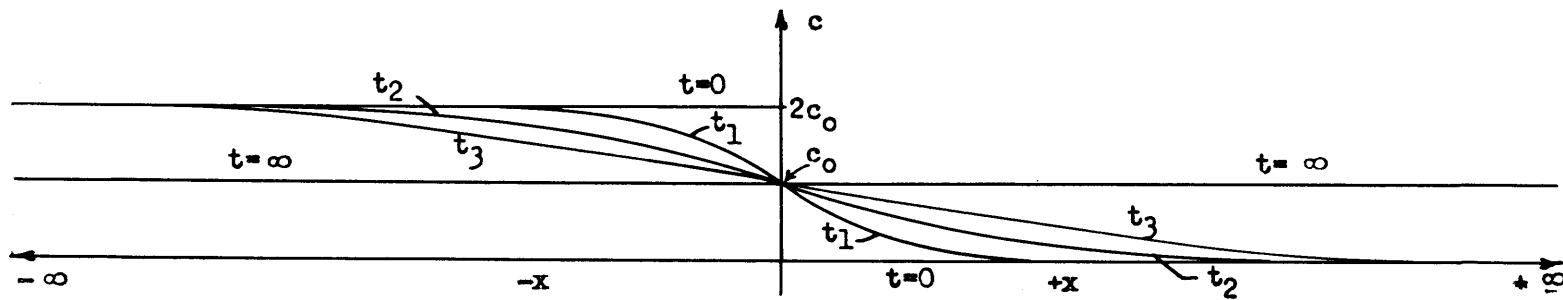
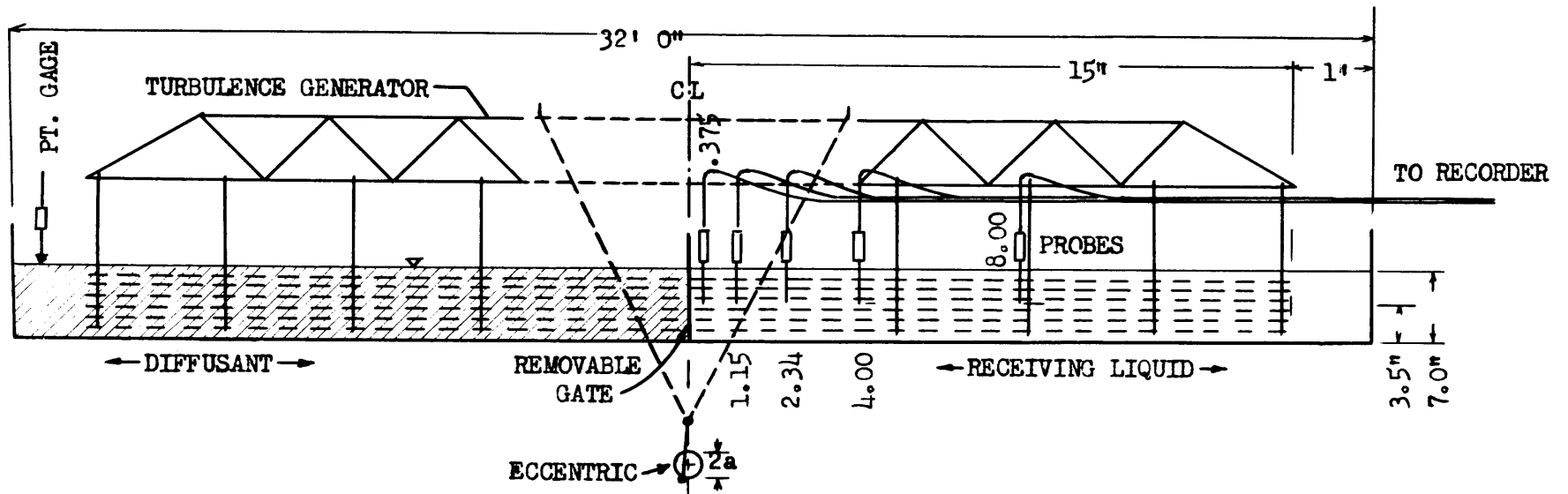


FIG. 10. SCHEMATIC DIAGRAM OF EXPERIMENTAL SET-UP FOR TYPICAL TIME-DEPENDENT DIFFUSION TEST SERIES.

After preparation of the solutions, a run was made as follows: at time $t = 0$ with the turbulence generator running at desired amplitude and frequency, the dividing plate was removed and the diffusion process began. Concentrations in the receiving liquid half of the flume were determined by direct probing where the diffusant was conductive or otherwise by sampling, at mid-channel and mid-depth for various distances from the original dividing interface, ($x=0$).

Detailed procedure for time-dependent diffusion tests.

The flume was filled up to a predetermined level with fresh water near room temperature, about two hours before the run was to begin. If conductivity measurements were to be made, the voltage stabilizer and Sanborn recorder were turned on at the same time to allow an ample warm-up time. Calibration solutions were prepared or placed in readiness where required.

The desired amount of diffusing substance (salt, dye or sediment) and, where needed, of substance used for specific gravity control in the receiving medium (sugar or salt) was weighed and (except in the case of the sediment) made up into stock solutions of equal volume. Where no specific gravity adjustment was to be made, a volume of fresh water equal to the volume of diffusant stock solution was placed in readiness.

The rubber sealing strips of the gate were coated with vacuum wax to ensure water tightness and the gate placed in position in the gap between the screens at the center of the flume. It was firmly forced against the sides and bottom and secured by means of clip angles and wedges. In controlled density tests, the prepared solutions of diffusant and of s.g. control substance were then introduced simultaneously at both ends of the flume while the turbulence producing mechanism was in motion. A certain degree of mixing with the fresh water took place in each half compartment while the quantities were introduced. About 1 to 1 1/2 grams of Methylene Blue dye were added to the diffusant, in cases where this was not already a dye. To ensure complete uniformity in each half compartment, the mechanism was stopped after introduction of the stock solution, allowing the partially mixed substance in both the diffusant and the receiving medium compartments to stratify until no more fluid motion could be observed (visible because of dye or density layers).

The mechanism was then set to oscillate at the desired frequency for the run and an ample time (4 minutes) allowed for the concentration to change from the stratified condition to uniformity with depth. At this point the gate was suddenly pulled, the recorder or the timer started at the same time, and the unsteady run proper commenced. The conductivity probes used for concentration determination in the tests where salt water was diffused, were switched in turn to the recorder, according to the following general plan. The probes nearest the original vertical interface were continuously alternately recorded for the first two minutes; thereafter each probe was recorded in turn every minute, until the end of the first 20 minutes of the run; then every two minutes for the next 20 minutes and finally every four minutes until the run was terminated.

The run was stopped, somewhat arbitrarily, before four times the time had elapsed in which it took the last probe ($x = 8$) to begin to register a concentration increase, i.e. the run was stopped approximately when a concentration increase would be likely to appear at the far end of the screens ($x = 15$ ft.) This time was about 65 minutes for the faster diffusion rates and several hours for the slower diffusion rates. However, it was considered unpractical and unnecessary to run for longer periods than one and a half hours. Each run thus yielded about 50 to 100 experimental points, in most cases about 80 were plotted.

In runs where there was practically no density difference between the diffusant and the water in the flume, a different technique had to be employed to ensure uniformity of the diffusant at the initial condition. The stock solution was introduced in small measured amounts uniformly with distance along the length of the diffusant solution of the flume while the screens were in motion and the gate was in place. A period of 20 minutes was allowed to elapse before the run was started, in order to even out any local concentration gradients.

For the dye and sediment (polystyrene spheres) tests, where samples of the diffusion process were taken only a limited number of samplings could be made during the initial stage of the run, so that a total of about 30 to 40 experimental points could be obtained per run. This number was adequate for a complete reconstruction of the diffusion process history.

Samples were drawn off by means of pipettes at uniform increments of t (at $t = 1, 4, 9, 16, 25, 36 \dots 100$ minutes) at six sampling stations ($x = 0, 1.15, 2.34, 4, 8, 15$ feet). Samplings were made with 30-second intervals about these time increments at all stations where the diffusant concentration had reached measurable values. These samples were analyzed by means of the spectrophotometer in the case of dye or drained, dried and weighed in the case of the sediment.

Methods for determining concentration.

The concentration distribution of diffusant in the receiving medium was determined by the following methods:

Where salt water was used as diffusant, the conductivity of the water was recorded in situ by means of conductivity probes located at mid-depth at various distances from the original interface. Since the probes could not be moved once the run has started, calibrations could only be made before and after the run so that the data reduction had to be based on calibrations made from two to three hours apart.

Where dye was used as a diffusant, concentrations of the samples taken were determined by means of a spectrophotometric analysis. A dye had to be chosen that follows Beer's Law of optical density and had an absorption band in the center of the visible range.

Three dyes were tried out - Methylene Blue, Methylene Violet and Rhodamine B, as these were locally available. Methyl Violet tended to adhere in a discoloring

film on the walls of the light absorption test cell, making cleaning difficult. With Methylene Blue, the absorption band was not ideally located, while Rhodamine B showed excellent agreement with Beer's Law and had an absorption maximum at 554 millimicrons, the center of the visible spectrum. The latter was, therefore, used for all quantitative dye tests. (However, Rhodamine B had a tendency to discolor the aluminum screens so that between tests a detergent and several rinses had to be used to remove all traces of the dye. For qualitative indication of the diffusion process in salt-fresh and salt-sugar and polystyrene sphere tests, Methylene Blue was used to color the diffusant, as this did not show any tendency to remain on the aluminum screens.)

The samples were transferred to 1 cm. thick absorption cells and the spectrophotometer scanning wavelength set at 554 $m\mu$ where the absorption was a maximum. The transmittance of the sample was compared to the transmittance of distilled water in an identical absorption cell (=100%). The transmittance of the sample at this wavelength was given within a few seconds by an indicator on the spectrophotometer, operating on an automatic servomechanism principle and the value was recorded for each sample. A complete spectrum curve was run for one fresh water sample, taken prior to the diffusion run, from the right hand compartment, as well as for one c_0 sample at $x = 0$, in order to verify that there was no foreign discoloring substance present in either the fresh water or dye solution and that the transmittance was determined at the correct wavelength.

Where the diffusant used was polystyrene sphere, 10 ml samples of the fluid were taken by means of pipettes. The spheres were sieved beforehand, only those retained between sieves with 30 and 50 meshes per inch were used. The bore of the pipettes was approximately 50% larger than the diameter of the largest sphere passing a 30 mesh sieve. Immediately after being taken, each sample was transferred to a filter paper cone, drained dried and weighed by means of a Sartorius DP2 Projection Balance, reading to 0.1 milligram.

VI ANALYSIS OF DATA

A. Power Input Tests

The calculation of the mean power input of the turbulence generating screens was made by averaging over a cycle the instantaneous hydrodynamic drag force multiplied by the instantaneous screen velocity as derived in Eq. [66]. Both the power input and the hydrodynamic force are shown in Plot 3. Since the hydrodynamic drag force F_D was approximately in quadrature with the forces due to spring deflection and inertia, the subtraction of the spring and inertia forces from the total measured connecting-rod force to determine F_D does not introduce a serious error as the hydrodynamic drag force is near the maximum. In the measurement of total force, the error due to the slope of the connecting rod (the force measured being too large by a factor equal to the secant of the angle between the connecting rod and the vertical) was negligible. The local drag coefficient, C_D^i , could therefore be estimated directly from the maximum drag force by Eq. [64]; or the average drag coefficient, C_D , from the mean power dissipation over a cycle by Eq. [66]. Since the only force that does work is the hydrodynamic drag force, and since this is in phase and of the same sign as the velocity at its point of application, the power dissipation is theoretically never negative at any part of a cycle. (Experimentally there is evidence of a phase lag between the maximum of the screen velocity and the drag force as seen in Plot 3, so that some work is done by the fluid on the screens at the points where the direction of motion reverses.)

The values of these two drag coefficients, C_D and C_D^i , computed in Tables II and III, agree closely with each other for all runs. It was also found that a small correction in the amplitude was necessary since the dynamic amplitude of the truss and screens was approximately determined to be slightly larger than the mean static amplitude, depending on the frequency. This correction improved the agreement in the drag coefficients. Table II summarizes the runs for the determination of power dissipation and gives the correlations with mechanical eddy viscosity, screen frequency and amplitude. The drag coefficient in steady flow is dependent on Reynolds number; however, the results indicate that for the complex flow pattern produced by the oscillating screens an average drag coefficient, C_D , may be obtained in order to represent the power dissipation over the entire range of the present experiments.

There exists a functional relation between the mean power dissipation into the fluid medium and the mechanical parameter, $a\sigma$, which is proportional to the turbulent intensity generated by the oscillating screens. An unique relation was found from all experimental data shown in Plot 4 as the following:

$$\begin{aligned}\bar{P} &= 0.55 a^3 \omega^3 \text{ (h.p.)} \\ \text{or} &= 303 a^3 \omega^3 \text{ (ft.lbs/sec.)}\end{aligned}$$

which is checked favorably with the result of Eq. [66] by taking the value of C_D from Table II.

TABLE II. ANALYSIS OF MEAN POWER INPUT, TESTS FOR DETERMINING C_D VALUES.

1	2	3		4	5	6	7	8
Series Run No.	σ (c.p.s.)	Static		$a^3\omega^3$	mean H.P.	\bar{P} (ft.lb./sec.)	$\frac{\bar{P}}{a^3\omega^3}$	C_D
		a (inch)	$a\sigma$ (ft-c.p.s.)					
2B	5.15	.238	.102	.263	.158	86.9	330	6.11
2B	4.84	.238	.098	.219	.131	72.0	329	6.10
2B	4.62	.238	.092	.190	.106	58.3	307	5.69
2B	4.28	.238	.085	.151	.089	49.0	324	6.00
2B	3.53	.238	.070	.085	.044	24.2	285	5.28
							ave.	5.84
3B-1	5.06	.232	.098	.235	.137	75.4	320	5.93
3B-2	4.56	.232	.088	.173	.111	61.1	353	6.54
3B-3	3.86	.232	.075	.104	.057	31.3	301	5.57
3B-4	3.68	.232	.071	.091	.053	29.1	320	5.93
							ave.	5.99
4-1	3.58	.428	.128	.516	.294	162	314	5.81
7-1	3.80	.432	.146	.777	.409	225	290	5.37
7-2	4.11	.432	.159	1.002	.598	329	328	6.08
7-3	4.74	.432	.186	1.596	.930	511	320	5.93
7-4	4.78	.432	.189	1.676	.888	488	291	5.39
7-5	4.75	.432	.190	1.573	.906	499	317	5.87
							ave.	5.74
5-1	2.39	.625	.125	.477	.273	150	315	5.83
5-2	2.34	.625	.122	.446	.255	140	315	5.83
							ave.	5.83
6-1	2.70	.628	.143	.732	.438	241	329	6.10
6-2	3.10	.628	.166	1.14	.550	302	265	4.91
6-3	3.47	.628	.189	1.66	.855	470	283	5.24
6-4	3.61	.628	.196	1.85	.950	522	282	5.22
							ave.	5.37

TABLE III. ANALYSIS OF MAXIMUM DRAG FORCE
FOR DETERMINING C_D VALUES.

1	2	3	4	5	6
Series Run No.	Amplitude a (in ft.)	$(F_D)_{\max}$ (in lbs.)	$a^2\omega^2$	C_D'	mean value of C_D'
3B-1	.0194	275	.381	5.68	5.86
3B-2	.0194	230	.311	5.82	
3B-3	.0194	166	.221	5.91	
3B-4	.0194	155	.202	6.04	
4-1	.0357	505	.643	6.19	6.19
7-1	.0385	615	.845	5.73	5.38
7-2	.0387	730	1.000	5.75	
7-3	.0392	915	1.366	5.28	
7-4	.0395	890	1.411	4.96	
7-5	.0390	890	1.352	5.18	
5-1	.052	440	.610	5.68	5.61
5-2	.052	425	.584	5.73	
6-1	.053	620	.812	6.01	
6-2	.0536	710	1.09	5.13	
6-3	.0543	985	1.40	5.54	
6-4	.0541	1060	1.506	5.55	

B. Time-Dependent Diffusion Tests

The analysis of the data is based on the diffusion equation for the case of one-dimensional diffusion in a semi-infinite medium in which a constant concentration is maintained at one boundary. For one-dimensional diffusion in a continuous medium with turbulence homogeneous in the direction of diffusion, it is usually assumed that the process can be treated as an analogue of molecular diffusion. One of the preliminary objectives is to investigate the validity of this analogy of the classical diffusion theory. Hence, the problem is to determine the turbulent diffusion coefficient and to investigate its correlation with the intensity of turbulence and with the density difference between the diffusant and receiving fluid.

1. Definitions of the Turbulent Diffusion Coefficients

(a) The turbulent diffusion coefficient or eddy diffusivity, $E_{x,t}$, is the diffusion coefficient due to turbulence only. The diffusion coefficient without subscripts, E , will be used to designate the special case in which E is independent of x and t . For this case the solution of the diffusion equation is given by Eq. [49].

$$\frac{c}{C_0} = 1 - \operatorname{erf} \frac{x}{2\sqrt{Et}}$$

(Here $2C_0$ is equal to the concentration of the diffusant at $x = 0$.) This equation is represented by a family of curves with E as a parameter (dotted lines) in Plot 8.

(b) The diffusion coefficient due to turbulence and gravity convection arising from a density difference between the diffusant and the receiving medium is termed the gross diffusivity, $E'_{x,t}$. As in the previous case, if the gross diffusivity is independent of x and t , it will be designated as E' . The gravity convection is caused by the effect of vertical density gradients in the diffusion process.

(c) An asterisk is applied to the diffusion coefficients, either E or E' , to indicate interpolated values of the diffusion coefficient obtained by plotting the experimental data on the family of curves of constant diffusion coefficient as shown in Plot 8. The experimental data, given by two typical runs (with and without density difference) in Plot 8, show that the diffusion coefficients interpolated in this manner decrease with an increase in the distance of the measuring stations. The apparent diffusion coefficients, E and E' , therefore, have no physical significance since the family of theoretical curves were obtained under the assumption that E was independent of both x and t . Since the field of turbulence is essentially homogeneous in the direction of diffusion, the reasons for the lack of agreement with the analogous molecular diffusion phenomena must be investigated in greater detail.

The experimental program can be divided into four phases according to the diffusant and receiving fluids, with or without initial density difference; namely,

- (i) Salt-water diffusing into fresh-water with the initial density difference of 1 ‰ or 2 ‰ (Salt-Fresh)
- (ii) Salt-water diffusing into sugar-water without density difference (Salt-Sugar)
- (iii) Dyed fresh-water diffusing into clear fresh-water without density difference (Dyed-Clear)
- (iv) Tracer-spheres in salt-water diffusing into salt-water without density difference (Tracer-Spheres)

Two amplitudes of the screen oscillation were used $a = 0.432''$ and $0.620''$ which are respectively smaller and larger than half the screen spacing ($0.50''$). The frequency varied from $\sigma = 1.97$ to 4.40 c.p.s. These are tabulated in Table IV from Col. 1 to Col. 5.

The data for each run of the time-dependent test were plotted as shown in Plots 5, 6 and 7 which describe both c/c_0 versus x with t as a parameter and c/c_0 versus t with x as a parameter. As a first attempt, the solution of the classical diffusion equation with E independent of x and t was checked by plotting the experimental data on the family of curves representing the classical solution of constant E , as a typical example shown in Plot 8 with c/c_0 versus $\frac{x}{\sqrt{t}}$ in arithmetic coordinates. The results do not satisfy the solution to yield a single curve with constant diffusion coefficient for each run. However, an apparent diffusion coefficient, as defined in (C), could be found. For any particular run, it is apparently independent of t but varies with the distance from the barrier as given in Table IV, Col. 6, 7 and 8. For further analysis, the diffusion coefficient is treated as a function of both x and t . A graphical method used to compute $E_{x,t}$ and $E'_{x,t}$ is described in the following paragraphs.

2. Method for Determining the Diffusion Coefficient as a Function of x and t

The results of all runs show the diffusion coefficient depending on x and t . This phenomenon could only be explained as a characteristic of the turbulent diffusion process which differs from the classical solution for constant E . An effort to understand the mechanism is necessary to treat the diffusion coefficient as a function of x and t . Since E is unknown function in the diffusion equation, the best way to find this function is by a graphical method from the experimental data as mentioned on p. 31. For Run 33 (with dye) the experimental values for concentration versus time were plotted on arithmetical coordinates (Plot 7). The experimental points for $x = 1.15, 2.34, 4, 8, 15$ feet were connected by smooth curves and values of c/c_0 for times $t = 250, 500, 1000$ to 6000 secs. were taken off and plotted against the respective values of x . Smooth curves c/c_0 versus x were drawn through these derived points. From the last set of curves, values were taken off for $x = 1, 2, 3, 5, 6, 7$ feet and plotted against the appropriate times on the same plot as the first set of curves. These plots yielded a complete reconstruction of the diffusion process (mid-depth concentration versus time and distance.) From the curves, values of $\frac{\partial c}{\partial x}$ and $\frac{\partial c}{\partial t}$ were taken off at one-foot intervals and at the above times. $\frac{\partial c}{\partial t}$ versus x was plotted and $\int_0^x \frac{\partial c}{\partial t} dx$ obtained by arithmetic integration for 0.5 ft. increments of x . The local turbulent diffusion coefficient E could be obtained (See Eq. [52], p.32):

$$E_{x,t} = \frac{\int_0^x \frac{\partial c}{\partial t} dx - \int_0^{\infty} \frac{\partial c}{\partial t} dx}{\frac{\partial c}{\partial x}}$$

A plot of $E_{x,t}$ thus obtained versus c for various times (Plot 9) showed that E was practically independent of time for the region $0 < x < 4$, and showed an inverse functional relationship of E with distance for all values of t , hence the values of E were averaged with respect to time. For $x > 4$ there was some divergence of the values of E , however, not much reliability could be attached to the values of E for $x > 4$ for reasons (i) the curves c/c_0 versus x were based on experimental data at $x = 4$ and $x = 8$ and the element of subjective interpretation enters in the drawing of a fair curve over this wide interval, (ii) the method breaks down where both the numerator and denominator of the equation become small: for $x > 4$ the value of $\frac{\partial c}{\partial x}$ is zero or small, except at large times, but for large times the de-

termination of $\int_0^{\infty} \frac{\partial c}{\partial t}$ is inaccurate as the tail of this integral for $x > 8$ has to be estimated.

The same method was subsequently used for analysis of Dye Runs 29, 27, 28 and it was found likewise that E varies approximately inversely with distance for small distances but remains practically constant with time (Plot 9). It was found that E at $x = 4'$ was considerably less than E_4^* (the apparent value E^* at $x = 4'$ (See Table IV.))

Runs 11 (Salt-Fresh) and 16 (Salt-Sugar) were re-analyzed by the above exact method in order to determine the values of the turbulent diffusion plus gravity convection (gross diffusivity) coefficient $E'_{x,t}$ and the eddy diffusivity $E_{x,t}$. Runs 11, 16 and 27 are all at the same frequency and amplitude ($a < \frac{S}{2}$) and offer a comparison (See top half of Plots 5, 6 and 7) between the behavior when a density gradient is present, Run 11 (Salt-Fresh), compared to no density gradient, Runs 16 (Salt-Sugar) and 27 (Dye).

Runs 24 and 25 at the same amplitude ($a > \frac{S}{2}$) and frequency were re-analyzed by the above method and together with Run 33 offer a comparison (bottom half of Plots 5, 6 and 7) between density effects, Run 25 (Salt-Fresh) and no density effects, Runs 24 (Salt-Sugar) and 33 (Dye) for amplitudes greater than half the spacing. The rate of variation of E_x and E'_x with respect to distance from the origin is large for $x < 1$ (ft) and it rapidly approaches a constant value. This phenomenon is worthy of additional investigation because it means that the classical solution can be applied for the major portion of the diffusion distance with the exception of a small distance near the origin.

3. Time-Dependent Diffusion as an Analogy of the Classical Diffusion Theory

The ultimate form of presenting the data obtained from the time-dependent diffusion tests is suggested by the above result that the diffusion coefficient is independent of x and t except near the origin. The solution of the classical diffusion equation with constant E can thus be satisfied by a modification of the diffusing distance, x , for the experimental data. After a shift of origin, x_0 , is applied to each run, the concentration c/c_0 , the modified diffusing distance, $x' = x - x_0$, and the diffusing time, t , satisfy the classical solution Eq. [49] for constant E . The method used to determine E and the shift, x_0 , from the data for each run is as follows:

A plot of the error integral versus its argument on arithmetic-probability coordinates yields a straight line with slope $\frac{2}{\sqrt{\pi}}$, see p. 31. If the error integral of $\frac{x}{2\sqrt{Et}}$ is plotted versus $\frac{x}{\sqrt{t}}$, a family of radial lines with slopes $\frac{1}{\sqrt{\pi E}}$ is obtained. Hence, after the shift of the origin, a plot of c/c_0 versus $\frac{x'}{\sqrt{t}}$ on arithmetic-probability coordinates should yield a straight line for all data for one run. The diffusion coefficient E or E' may then be uniquely determined from the slope for each run. The shift x_0 is obtained by trial and error in order to locate all points for measuring stations, greater than x_0 , on the same straight line.

It should be noted that $x' = 0$ is not the origin (the initial interface of diffusant and receiving fluid) any more, thus the straight line passing all experimental points does not necessarily meet $c/c_0 = 1.0$ at $x'/\sqrt{t} = 0$.

A plot of c/c_0 at $t = 250, 500, 1000$, etc. versus $\frac{x'}{\sqrt{t}}$ for a typical run (Run 33) on arithmetic-probability paper yielded almost a perfect lining-up of all points along a straight line (Plot 13 b). The slope of this line gave a good agreement with the asymptotic value of $E_{x,t}$ in Plot 9 as described before. Similar plots were made for the rest of runs, and E and E' were found in the same manner. The results of all runs for E , E' and x_0 were tabulated in Table IV at the end of this Section.

As a result of shifting the origin the experimental data could be satisfied by the solution of the classical diffusion equation with E or E' independent of x and t , but E and E' are dependent on the turbulent intensity of the diffusing medium. As derived in Eq. [68] and [66], the mechanical eddy viscosity in the direction of diffusion is proportional to the $a\sigma$ and hence the mean energy dissipation per unit volume. Plot 15 with E and E' versus $a\sigma$ summarizes the results of all runs and shows the correlation of the diffusion coefficient with the intensity of turbulence. The effect of a density difference between the diffusant and the receiving fluid is designated by the parameter $\Delta\rho/\rho$.

$HP = 136(\sigma)^3$

TABLE IV. SUMMARY OF RESULTS OF TIME-DEPENDENT TESTS

1	2	3	4	5	6	7	8	9	10	HP	HP	
Category	Run	a ampl. (inch)	σ freq. (cps)	$\sigma\sigma$ (ft/sec)	E'*, E* at x =			X_0 (ft)	40 slugs E or E' (ft ² /sec)		HP	HP
					2.34'	4'	8'					
I Salt-Fresh 1%	23	.432	2.66	.0956	.0040	.0030	.0023	.5	.0035			
	11	.432	2.95	.105	.0059	.00415	.0035	.55	.0032			
	22	.432	3.26	.117	.00443	.0033	.00287	.4	.0033			
	18	.432	3.67	.132	.00345	.00345	.00345	0	.0036			
	20	.432	4.40	.158	.0033	.0033	.0033	0	.0029			
	26	.620	3.33	.179	.00332	.0030	---	.2	.0028			
	2%	13	.432	2.95	.106	.0109	.0067	---	.8	.0067		
II Salt-Sugar	19	.432	2.22	.080	.0023	.00156	.00114	.5	.0017	.00113	.070	
	16	.432	2.94	.106	.00293	.0023	.00193	.3	.0044	.0020	.16	
	17	.432	3.70	.133	.0029	.0024	---	0	.0084	.00255	.32	
	24	.620	2.53	.131	.00235	.00169	---	.15	.0077	.0017	.31	
III Dyed-Clear	29	.443	1.97	.073	.00241	.00143	.00111	.8	.0013	.0008	.053	
	27	.443	3.01	.111	.00392	.00213	.00165	.45	.0047	.0016	.19	
	28	.443	4.13	.153	.00641	.00365	.00257	.6	.0122	.0023	.49	
	32	.443	2.99	.110	.00457	.00264	.00193	.8	.0045	.0015	.18	
	33	.645	2.08	.112	.00305	.00188	.00165	.8	.0048	.0011	.19	
IV Tracer Spheres	30	.443	2.96	.110	.00338	.0025	.00199	.5	.0045	.0019	.18	
	31	.443	3.05	.113	.0065	.00263	.00188	.7	.0052	.0016	.20	

VII DISCUSSION OF RESULTS AND CONCLUSIONS

A. Summary of the Results of Time-Dependent Tests

(See definitions of the diffusion coefficients, page 72)

(i) Salt-water diffusing into sugar-water with no density difference between the diffusant and receiving fluid.

- (a) $E_{x,t}$ is found to be independent of t but dependent on x , the distance from the origin. E_x decreases rapidly with increasing x for $x < 1'$ and approaches a constant value for $x > 2'$ for each run as shown in Plot 9.
- (b) E is independent of t and x' , the modified diffusing distance by shifting the origin a small distance x_0 . The plots of c/c_0 versus $\frac{x'}{\sqrt{t}}$ yielding E are shown in Plots 10 and 11 for typical runs without density difference between diffusant and receiving fluid.
- (c) E is linearly proportional to the turbulent intensity or $a\sigma$ as shown in Plot 15.

(ii) Salt-water diffusing into fresh-water with density differences of 1% and 2%. The results (a) and (b) of case (i) are also valid for E' for the density difference of 1% and 2% as shown in Plots 13 and 14. A few additional findings are:

- (a) For the same turbulence level, $a\sigma$, E' is found to be larger than E , depending on the magnitude of the density difference. The differential value of $E' - E$, which is due to gravitational convection, decreases with increasing turbulence level and approaches zero at large values of $a\sigma$ as shown in Plot 15.
- (b) Run 15, with $a = 0.432''$, $\sigma = 2.92$ c.p.s. and $\frac{\Delta\rho}{\rho} = 1\%$, made with probes placed at depths smaller and greater than the mid-depth, confirmed the suspicion that the concentration varied with the depth. The variation, at $x = 1.15'$, over the central 40% of the depth was from 33% of the mid-depth concentration at $t = 1$ min. down to 2.5% at $t = 22$ min. This vertical concentration gradient $\partial c/\partial y$ increased with increase of $\partial c/\partial x$ and became significant when the turbulent intensity was small and negligible when the turbulent intensity was sufficiently large. The presence of a vertical density gradient causes a convective circulation and alters the homogeneity of the generated turbulence both in the vertical and longitudinal

directions. The whole process is thus not one-dimensional turbulent diffusion but a two dimensional one with convection due to the vertical density gradient.

(iii) Dyed fresh water diffusing into clear fresh water without density difference between the diffusant and receiving fluid. In diffusion processes it is sometimes found that the diffusant is non-conservative, i.e. it is absorbed or combines chemically with the receiving medium so that the mathematical model with a constant concentration at the origin (the initial interface) is changed. As a consequence of the displacement of constant concentration point, the solution of the classical diffusion equation of constant E , Eq. [49], is no longer valid with respect to the original coordinates which implies the constant concentration at the origin. In order to clarify this point, tests of the combinations other than "Salt-Fresh" and "Salt-Sugar" were investigated. The method of sampling and analysis for dyed-fresh water diffusing into clear fresh water has been described in Chapter V.

Three runs were made at frequencies of 3.01 (Run 27), 4.13 (Run 28) and 1.97 (Run 29) and an amplitude of 0.445", using Rhodamine B dye ($2 C_0 = 8$ ppm) as diffusant. The results confirmed those of case (i), i.e. the diffusant and receiving fluid are conservative and a small shift of the origin gave constant values of E as shown in Plots 10, 11 and 12.

(iv) Tracer spheres as diffusant in receiving fluid of same specific gravity as spheres. To ascertain whether molecular diffusion may be assisting the turbulent diffusion process by providing the cross-eddy diffusion in regions where the turbulent concentration fluctuations are large (i.e. at the zone of steep longitudinal concentration gradients $c' = 1 \frac{dc}{dx}$, $\frac{dc}{dx}$ being large where x is small) an experiment was designed to prove or disprove this presumption.

If molecular diffusion is responsible for a distance variation of diffusion coefficient, the molecular diffusion could only be effective over small distances, such as in the smallest scale eddies present. Molecular diffusion effects on large scale transfer may be eliminated if the diffusant is not capable of concentration gradients or fluctuations over very small distances. A sediment, where the particles are far apart compared to the microscale of turbulence, would not be capable of rapid concentration changes at a sampling point since the concentration is determined by the mean particle position in a finite volume over a finite time.

A sediment was available of sp. gr. 1.053, in order to assure that the sediment would act as if it were a marked particle of fluid, the specific gravity of the water had to be adjusted to a value of 1.053 by adding salt. Salt was used since it has a small effect on the viscosity or molecular self-diffusion coefficient of water at the concentration required.

It was observed that the particles had no fall velocity when suspended in the salt solution and after shaken up, they stayed in place for hours (though after a day or two they were observed to have coalesced into small groups, dispersed throughout the liquid). It was found that a 10 ml pipette with 10 ml bulb was an adequate sampling device so long as the particles could freely pass through the opening, the particles acting as if part of the fluid. Two runs were made, each using a 2 kg sample of "retained between 30 and 50 meshes per inch" Koppers polystyrene spheres. The particles were volumetrically divided into 16 equal batches which were introduced uniformly with distance in the left-hand end of the flume when the mechanism was operating. The results again confirmed those of case (i) "Salt-Sugar" and case (iii) "Dyed-Clear" as shown in Plot 12.

B. Observations on the Irregularities of Boundary Conditions

(i) Uniformity of the Amplitude of the Oscillating Screens

Since it was observed that the diffusion coefficients without a shift of the origin are somewhat dependent on x , even for runs without density differences, a check of the amplitude of the screens along the flume was made. The total vertical displacement of the screens was measured at various distances from the center of the flume for a full range of frequencies. The motion of the truss was also recorded at various distances. It was found that under static pull the amplitude (spring deflection and flexural deflection of truss) was 17% greater at the center than at the ends. When the truss is in motion, however, the static force is reduced due to inertia, and the dynamic amplitude could be considered within 2 percent of uniformity. The greatest deviation of E_x or E'_x from constant E or E' occurs within 1 foot from the initial interface; hence, this is unlikely due to the non-uniformity of amplitude observed in the above tests.

(ii) Low Turbulent Intensity near the Bottom of the Flume

During tracer sphere tests, there is indication that the turbulence near the bottom of the flume was relatively small and that considerable migration of the particles occurred. In Run 30, the mean position of the screen was lowered 1/4" relative to the flume bottom and the turbulence close to the flume bottom was considerably increased but in viewing the result for the turbulent diffusion coefficient there was no detectable effect due to this adjustment.

(iii) The Conditions at the Ends of the Screens and the Gap between Screens at the Origin

It was also observed that there was a considerable in- and outflow around the ends of the screens and through 1/2" gap at the origin, left by the removal of the vertical partition. Run 31, in which the end baffles

were installed, did not show any effect on the diffusion coefficient.

Run 32 was designed to check the effect of the gap in the screens at the origin. Immediately after the run commenced and as soon as the gate was removed a section of the screens was pulled over to close the gap. No effect on the diffusion coefficient was observed.

These observations suggest that the minor defects in the boundary conditions are not responsible for the variations of E and E' near the origin. One factor which may have an affect is the disturbance caused by pulling the vertical partition.

C. Discussion of Results

1. One of the objectives of this investigation was to determine whether the classical diffusion theory may be used to describe turbulent diffusion in the one-dimensional case. The diffusion coefficient was found to be independent of diffusion time but dependent on the distance, x , in the immediate vicinity of the origin. This phenomenon can partly be explained in the light of the existing theory of turbulent diffusion. In the statistical theory of turbulent diffusion, the process can be postulated in two parts depending on whether the diffusion time is much larger than or smaller than the Lagrangian scale of turbulence, L_h . (see Sec. III on p. 41).

(i) Large diffusion times

In the case of the diffusion time being much larger than the Lagrangian scale of turbulence, the turbulent diffusion process can be treated by the classical diffusion theory. The results of the diffusion coefficients obtained by the graphical analysis and shown in Plot 9 indicate that the process of turbulent diffusion is represented by a solution of the classical diffusion equation with variable E_x as follows:

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left(E_x \frac{\partial c}{\partial x} \right)$$

E_x varies rapidly only near the origin and approaches a constant value within a short distance. The characteristic of E_x indicates that the experimental results for large distances from the origin may satisfy the statistical theory of turbulent diffusion for $t \gg L_h$. Considering a region of large x , mathematically, it is possible to exclude the region near the origin where E_x varies by a transformation of the diffusion distance, x , as follows:

$$x' = \int_0^x \sqrt{\frac{E}{E_x}} dx$$

With a shift of the origin, $x_0 = x - x'$, the classical diffusion equation

for constant E is obtained in the transformed coordinate, x' , as

$$\frac{\partial c}{\partial t} = E \frac{\partial^2 c}{\partial x'^2}$$

which will yield the solution as Eq. [49] with respect to x' and t . Since E_x or E'_x was computed graphically from the experimental data, a practical way to determine this transformation is by the trial and error method as described in the last section. This treatment is justified since after the transformation (the same shift of the origin applied to the data of all measuring stations in any run) a constant diffusion coefficient could be obtained. Evidently, this region corresponds the diffusion time being much larger than the Lagrangian scale of turbulence in the diffusion process. Hence, the experimental data analyzed in this manner yields the diffusion coefficient as a function of the turbulence level and density difference between the diffusant and receiving fluid.

In case of large diffusion time in the region near the origin, the above analysis cannot be applied because the magnitude of the shift has about the same order of magnitude of x , within which E_x or E'_x varies. The shifts of the origin for all runs were listed in Table IV. The average shift for all runs is 0.4 feet and varies between 0 and 0.8 feet, it is unlikely that any correlation for the shift can be found from the present experimental data. The scattered values of the shifts indicate that the disturbance caused by the removal of the vertical partition may be in some way responsible.

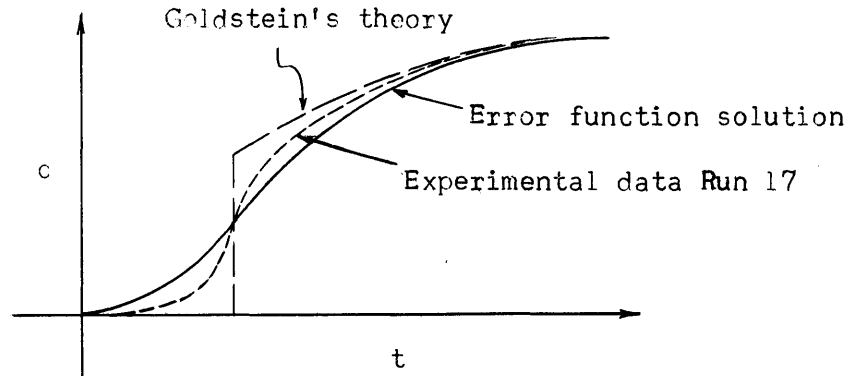
(ii) Small diffusant times

In the case of the diffusion time being smaller than the Lagrangian scale of turbulence, the classical diffusion theory cannot be applied. The corresponding region in this case is the immediate vicinity of the origin. The difference between the Goldstein diffusion theory (see Sec. III on p. 37) and the classical diffusion theory was examined. Significant differences should exist only in so far as $Vx/2E < 8$, where V is the migration velocity, hence $x < 16E/V$. The migration velocity, V , could be estimated from the test by observing the first rise of the concentration at a measuring station close to the origin at time, t , after the test started, since $V = x/t$. Taking some typical values:

$$\begin{aligned} E &= 0.0025 \text{ ft}^2/\text{sec} \\ V &= 0.04 \text{ ft}/\text{sec} \end{aligned}$$

from Run 17, where the first rise of the concentration was observed at $t = 9$ seconds for $x = 0.375$ ft. For this case the value of $16E/V$ is approximately one foot. Hence the difference between these two theories is likely to show up in the region $x < 1$ ft, i.e. only at the first probe location ($x = 0.375'$). An analysis of the typical data at this distance for Run 17 indicated no perceptible concentration rise for the few seconds, then a

substantial rate of increase lasting a few seconds, then a distinct reduction in slope asymptotic to $c/c_0 = 1$. As shown in the following sketch, the deviation from the Goldstein's prediction, that the concentration has a step-increase for $t < x/V$ and gradual changes over to the error function relationship, is believed to be due to the rounding effect of the statistical properties of the turbulence.



The conclusion is that the difference between Goldstein's diffusion theory and the classical diffusion theory is significant at a smaller distance x as compared with the region in which E_x varies with x , hence, it is only partially responsible for the dependency of E on x .

2. A further objective of the experimental investigation was to determine the effect of density differences between the diffusant and receiving fluid as expressed by gross eddy diffusivity, E' . It is noted, in Plot 15, that the diffusion coefficient without density difference, E , is linearly proportional to the turbulence level $a\sigma$ or the mechanical eddy viscosity. This process is a true one-dimensional diffusion with turbulence homogeneous in the direction of diffusion since there are no density gradients horizontally or vertically.

The effect of density differences can be divided into three categories with the turbulence level, $a\sigma$, and $\Delta\rho/\rho$ as parameters, as shown in Plot 15:

- (i) A region, in which both $\partial\rho/\partial x$ and $\partial\rho/\partial y$ are important, must exist at low turbulence intensities. In this case the two liquids tend to stratify and the one-dimensional analysis breaks down. This case is outside of the scope of the present experimental investigations.
- (ii) A region, in which the longitudinal convection as indicated by $\partial\rho/\partial x$ is important compared with the turbulence level. In this case $\partial\rho/\partial y$ is small so that the one-dimensional analysis may be considered as a reasonable approximation. In this region, the gross eddy diffusivity E' , is larger than E and depends on $\Delta\rho/\rho$; but the difference, $E' - E$, diminishes as $a\sigma$ increases.
- (iii) A region, in which the effect of $\partial\rho/\partial x$ on the diffusion coefficient is

negligible as compared to the diffusion due to the high turbulence levels. The curve for $\Delta\rho/\rho = 1\%$ approaches the straight line of $\Delta\rho/\rho = 0$ at a turbulence level, $a\sigma$, approximately equal to 0.2 ft/sec. It is probable that for larger density differences the approach would occur at correspondingly higher turbulence levels.

D. Discussion of Results in Relation to other Experimental Turbulent Diffusion Tests.

It was of interest to compare the findings of the present investigation with experimental results of others.

In the fields of oceanography and engineering fluid mechanics, it was generally evident that the investigators tacitly assumed that the Fickian Laws of Diffusion also hold for turbulent diffusion. In cases where experiments yielded variable diffusion coefficients an empirical relationship was usually assumed and ascribed to a variation in turbulence intensity, etc. In turbulent diffusion such a procedure may be in error, as pointed out by Frenkiel (C-3) since the turbulent diffusion equation involves an eddy diffusivity that is not a constant even for homogeneous, isotropic turbulence, but is a function of the Lagrangian time correlation parameter L_h . For $t \gg L_h$ the value of the eddy diffusivity attains a constant value and the classical diffusion differential equation (for constant E) may be applied.

On the other hand, in the fields of aerodynamics, meteorology and transfer operations in chemical engineering, the diffusion of material, momentum or heat across the relatively short boundary layer distances becomes very critical and in these fields the theoretical and experimental study of turbulent diffusion has been carried to a more advance state of detail and refinement. The classical diffusion theory is generally found to be inadequate to describe, for example, the heat transfer through a boundary layer in turbulent air flow across a flat plate. The nearest approach to the present ($\Delta\rho = 0$) investigation of diffusion due to turbulence, where the fluid elements have a mean velocity zero or small compared to the instantaneous fluctuations (a state known as "free turbulence", i.e. turbulence not associated with shear stresses due to velocity gradients) is found in the literature on diffusion through porous beds.

Experiments on diffusion in turbulent packed fluid beds yielded good experimental confirmation of Taylor's and Goldstein's equations. Wicke and Trawinski (G-1) measured transverse diffusion of mass and heat in a water fluidized packed bed. Mean square displacements from a point source of injection were obtained for heat transfer by means of warm water and for mass transfer by means of HCl injection. The diffusion coefficients obtained were found to increase with mean through flow rate (that is, with eddy viscosity), as would be expected, and moreover was found to decrease with increasing distance from the source, a result which is in agreement with the present $\Delta\rho = 0$ time-dependent tests. Wicke and Trawinski also found that the heat transfer coefficient was slightly greater than the mass transfer coefficient, but ascribed this as possibly due to the fact that the conductivity of the particles in the packed bed can aid the heat transfer but not the mass transfer of tracer.

Regarding the proportionality of E with ϵ in shear-flow fields, a look at the results of Longwell and Weiss (G-2) may be of interest. They studied radial transport of liquid (fuel) droplets in ducted turbulent air streams. They found that the mass transfer process was adequately described by the classical diffusion equation and that the lateral transfer coefficient (eddy diffusivity) was uniform over the duct cross section, but varied with distance in the direction of flow. They found with volatile droplets a 20 per cent increase in E with doubling of the distance from the source along the direction of flow which may be ascribed, however, as possibly due to growth of turbulence in the duct with distance.

E. Summary of Conclusions

1. The mean rate of energy dissipation by the turbulence generating screens was measured and correlated with the product of screen amplitude and frequency, $a\delta$. The product, $a\delta$, therefore defines the turbulence level of the liquid and is proportional to the eddy viscosity. It is concluded that the measurement of turbulent energy dissipation is a useful means of describing a state of turbulence.

2. The classical diffusion theory was found to be adequate to describe the one-dimensional, time-dependent, turbulent diffusion in a liquid with a constant concentration applied at one boundary ($x = 0$). In this case the classical diffusion equation is

$$\frac{\partial c}{\partial t} = E \frac{\partial^2 c}{\partial x^2}$$

and the solution, for the above boundary condition, gives the concentration of the diffusant as a function of distance and time as

$$\frac{c_{x,t}}{c_0} = 1 - \operatorname{erf} \frac{x}{2\sqrt{Et}}$$

3. For each test, a small shift in the diffusion distances (averaging 0.4 ft for all runs) yielded good agreement between the experimentally observed concentrations and the classical diffusion theory, with the diffusion coefficient independent of the modified diffusion distance and the diffusion time.

4. One of the primary objectives of the research program was to determine the effect of small density differences, between the diffusant and receiving fluids, on the gross longitudinal diffusion. Tests were made for the case of no density difference by the following means:

- (i) [diffusant, marked with dye] [fresh water receiving fluid].
- (ii) [saline diffusant] [sugar solution receiving fluid of same density as saline solution].

- (iii) [saline diffusant containing small plastic spheres in suspension (zero-fall velocity)] - [same saline solution, receiving fluid without spheres].

Tests were also made for the case in which a difference in density existed between the two fluids:

- (iv) [1% saline diffusant] - [fresh water receiving fluid].

- (v) [2% saline diffusant] - [fresh water receiving fluid].

- (a) For the diffusant and receiving fluid of equal density, the turbulent diffusion coefficient or eddy diffusivity, E , is linearly proportional to the turbulence intensity $u\sigma$. The correlation is therefore one of constant Schmidt number.
- (b) For the diffusant having a density one-percent greater than the receiving fluid, the gross turbulent diffusion coefficient, E' , for the same turbulence level, may be as much as 100 percent greater than for the case of zero density differential. The effect of the gravitational-convective mass transfer is expressed by the gross diffusion coefficient minus the diffusion coefficient due to turbulence only, $E' - E$. The quantity, $E' - E$, is approximately inversely proportional to the turbulence intensity (Plot 15). As the turbulence level approaches zero the two fluids tend to stratify and the one-dimensional analysis is invalid. At high turbulence levels, $E' - E$ tends to zero and the diffusion process is identical with (a). The results for the 2% density difference indicate $E' - E$ values much larger than for the 1% case.

5. For very small diffusion times, the existence of a non-Fickian diffusion process has been confirmed. The process is similar to the "telegraph equation" theory of turbulent diffusion proposed by Goldstein.

6. In the vicinity of the origin ($x = 0$) the diffusion coefficient varies with distance even for large diffusion times. For small diffusion times the departure from the classical theory is explained as in (5) above. For large diffusion times, the departure should become negligible according to the various theories of turbulent diffusion in a field of homogeneous turbulence. Hence, the fact that the diffusion coefficient is distance-dependent near the origin is not completely explained. It is suspected, however, that there is some effect of disturbances caused by removal of the vertical barrier separating the diffusant and receiving fluids at $t = 0$.

7. From the standpoint of waste water disposal in rivers or estuaries, the deviations from the classical diffusion theory under 5 and 6 are of minor importance. Exact boundary conditions and homogeneous turbulence are non-existent in nature. The experience gained in this fairly abstract study of turbulent diffusion has opened new approaches to the application of diffusion theory to waste water disposal. In particular, the investigation begun here is being carried forward to a study of steady-state diffusion phenomena in an idealized estuary or river.

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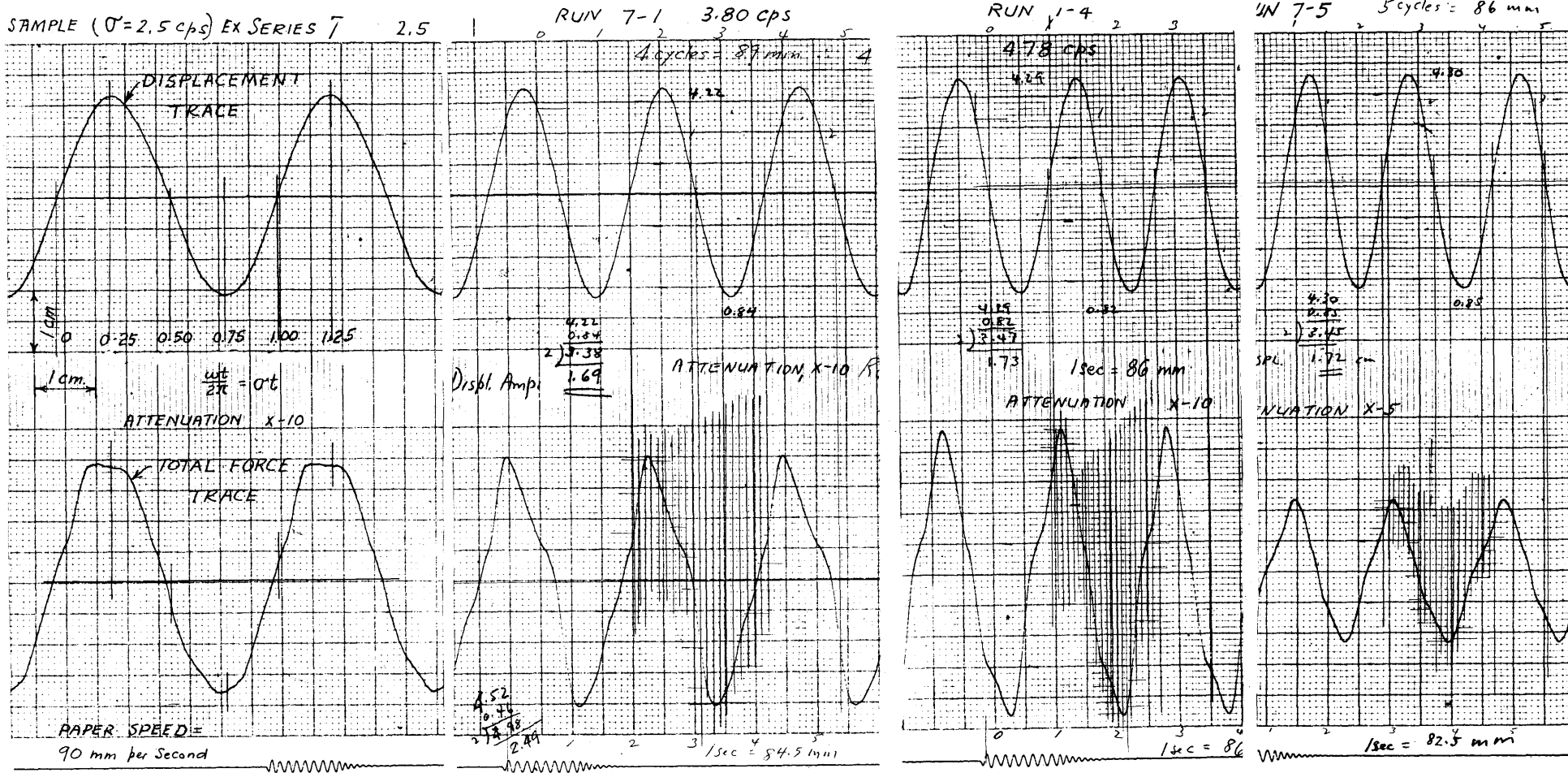


Fig. 11 Displacement and Total Force Traces taken during typical series of runs for determining power dissipated by turbulence generating mechanism.

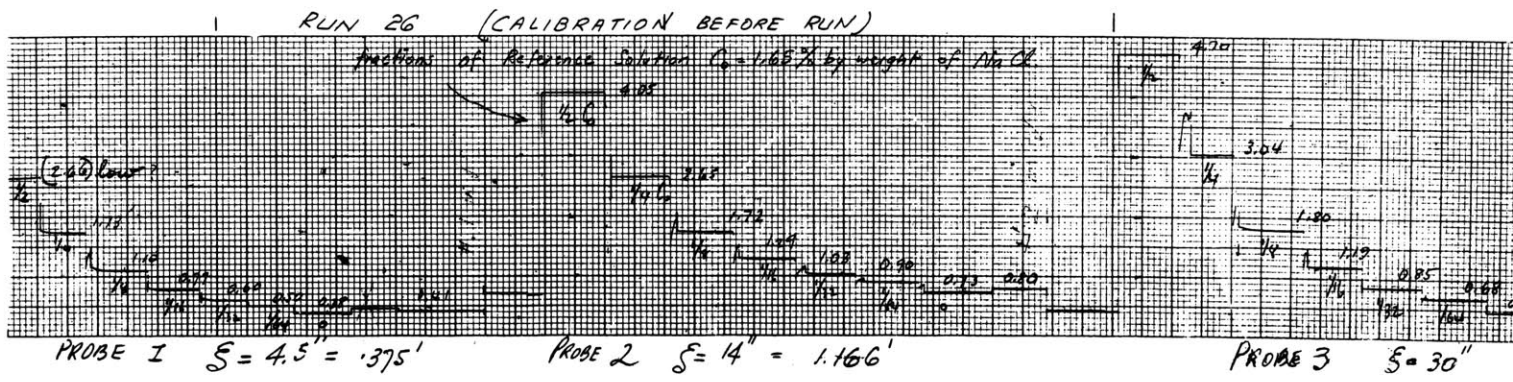


Fig. 12(a) Calibration record for conductivity probes, taken before typical time-dependent test

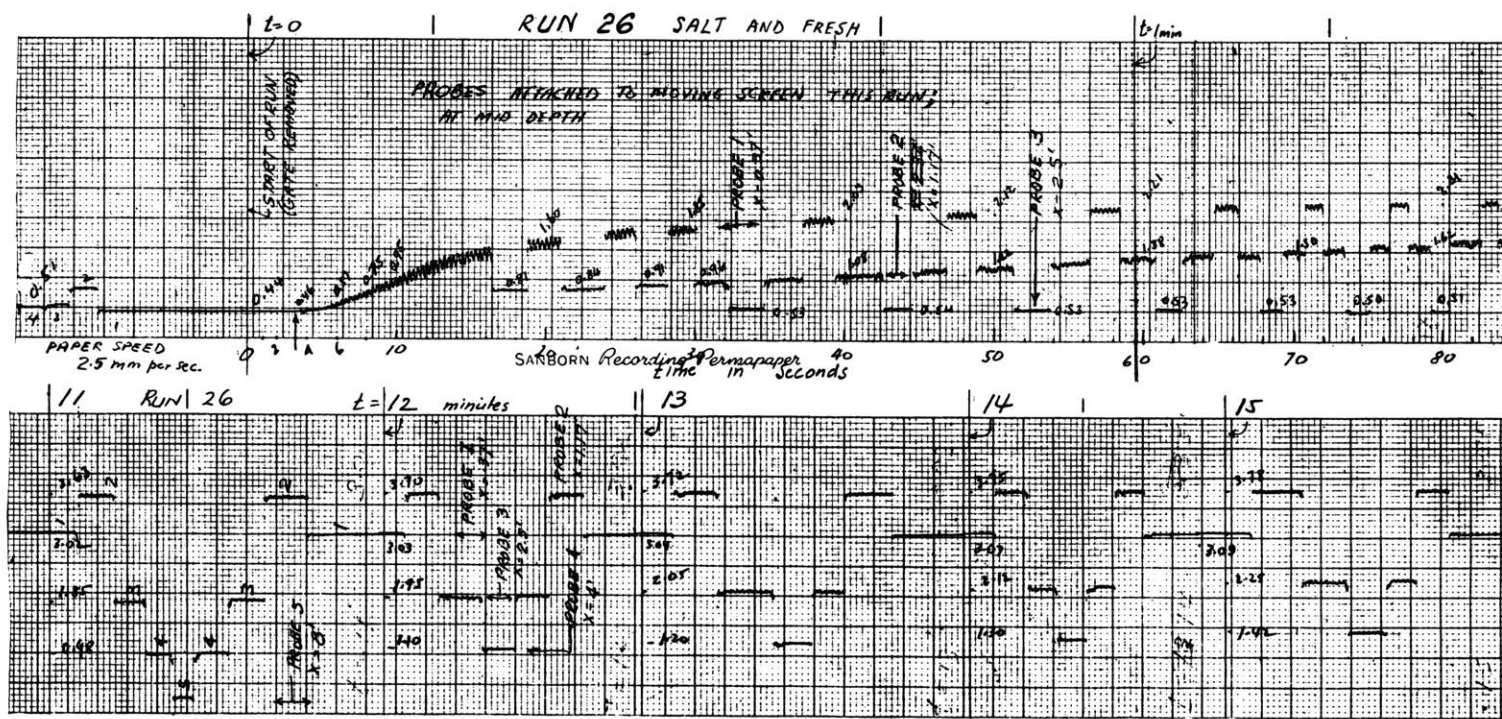
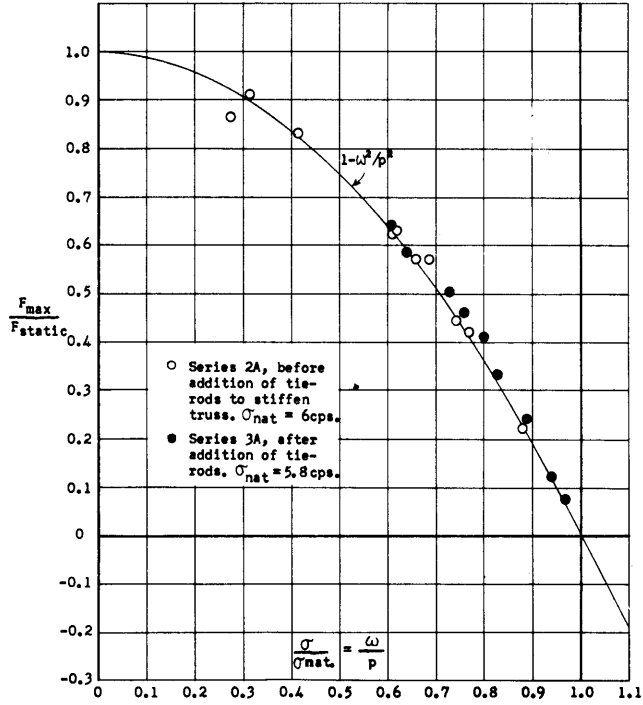
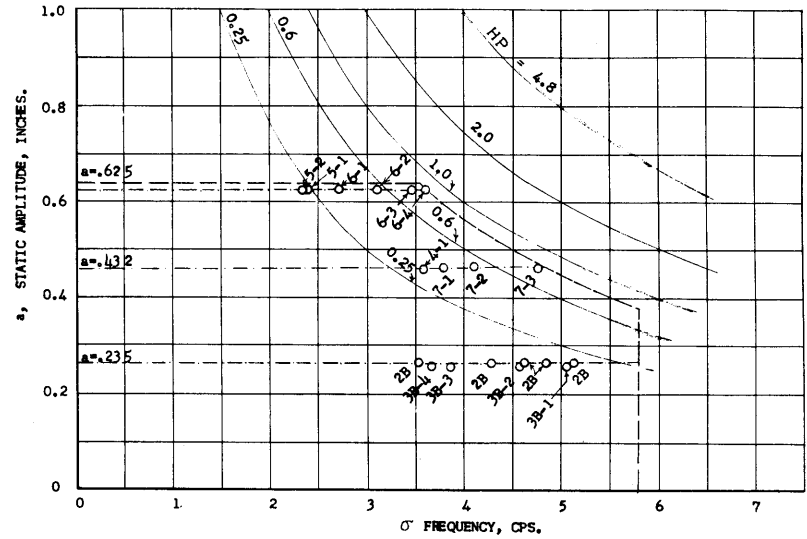


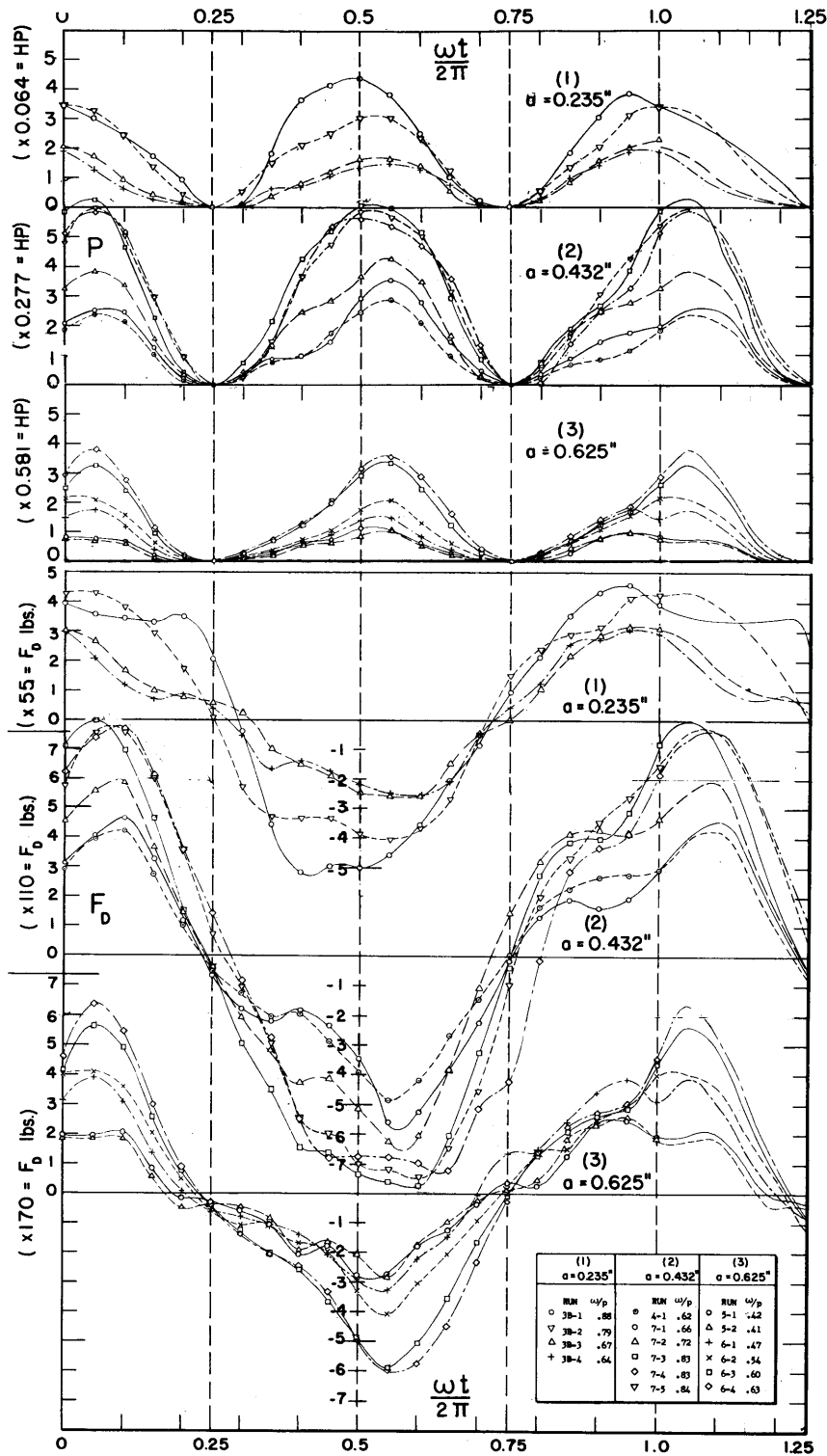
Fig. 12(b) Portions of record of conductivity probe output during a typical time-dependent diffusion test. Sequence from the same test for which calibration shown in Fig. 12(a)



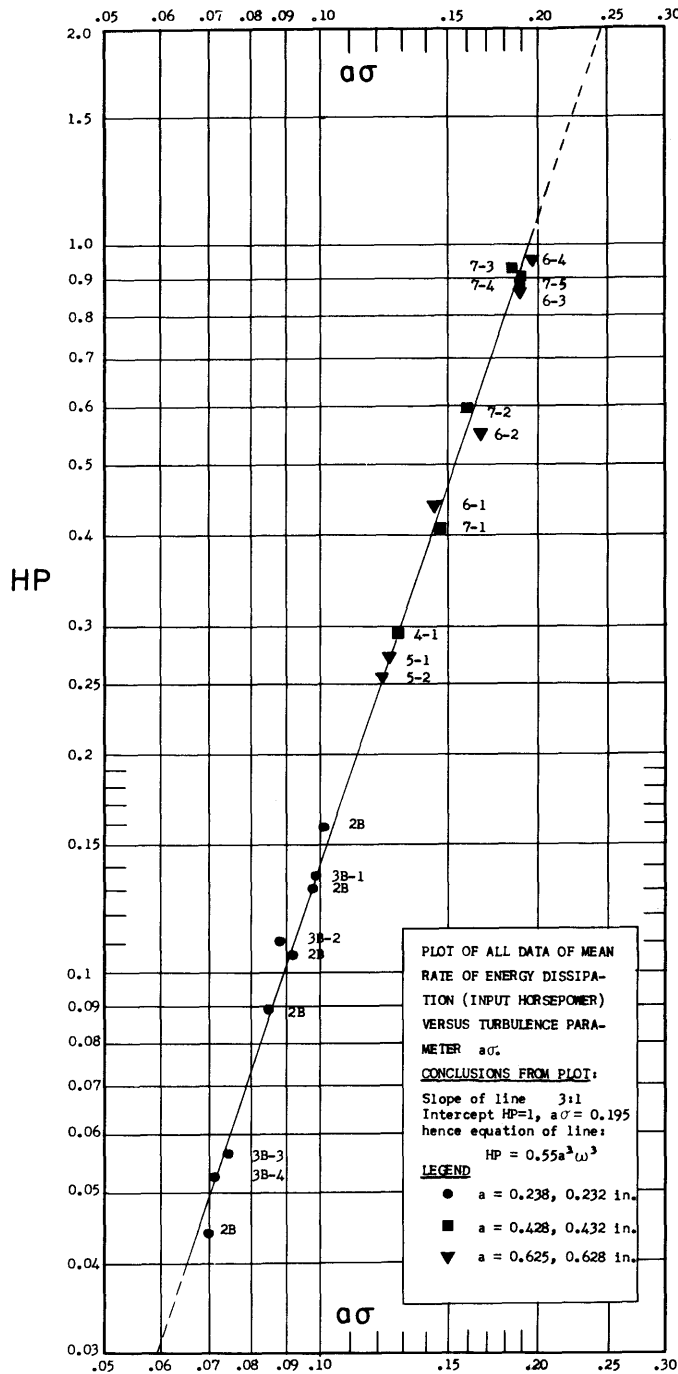
Plot 1. Force versus frequency for undamped vibration of truss with screens, determination of natural frequency.



Plot 2. Range of operation (amplitude and frequency) of turbulence generator with screens submerged. Lines of equal power input shown solid. Dashed lines show practical limits of operation.



Plot 3. Distribution of hydrodynamic drag force (F_D), and instantaneous rate of energy dissipation over a cycle for three amplitudes of oscillation over a representative range of frequencies.

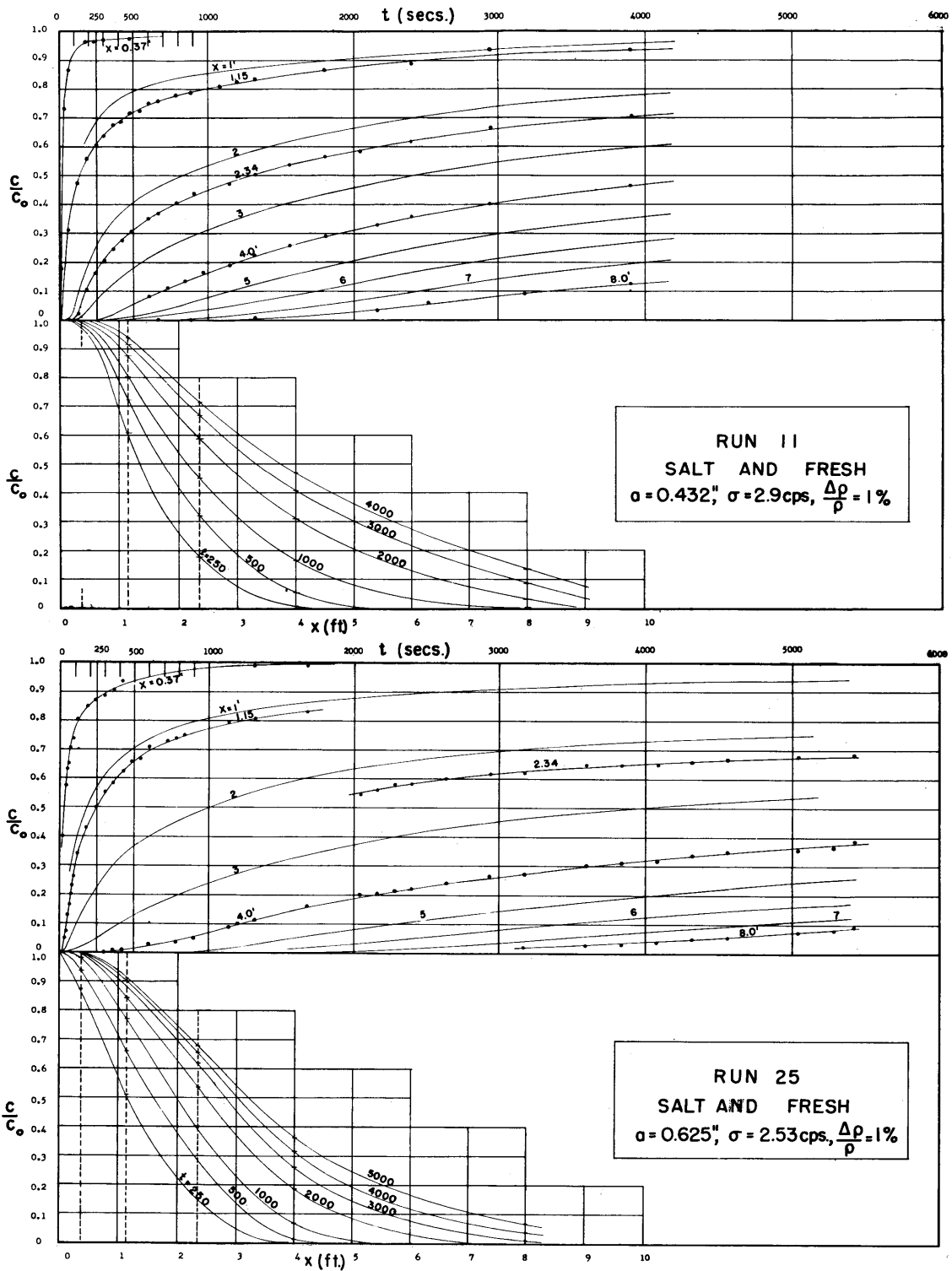


$\sigma = \text{rev per sec}$
 $\omega = \text{rad per sec}$

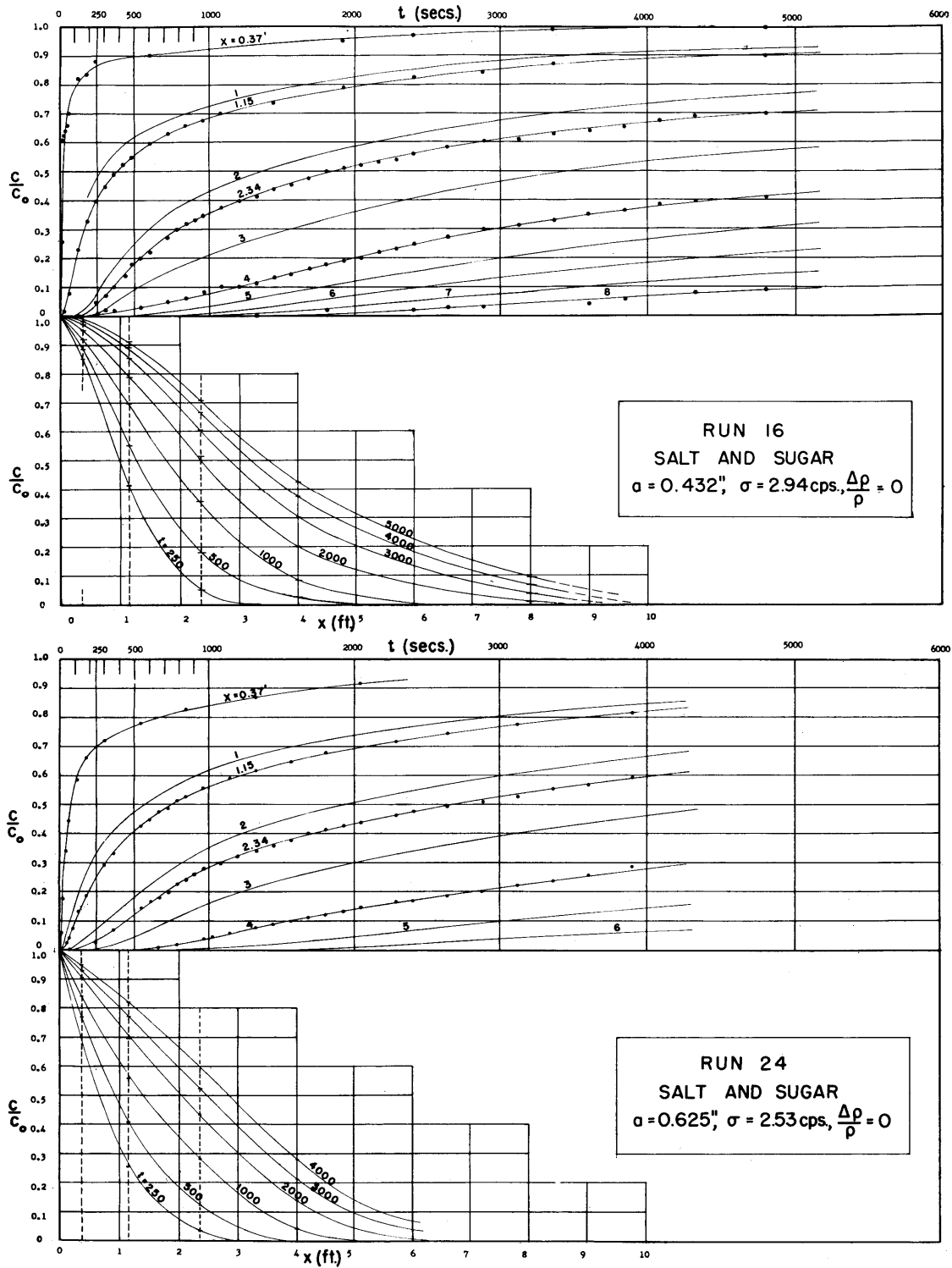
$\omega = 2\pi\sigma$

$(2\pi)^3 = 247$

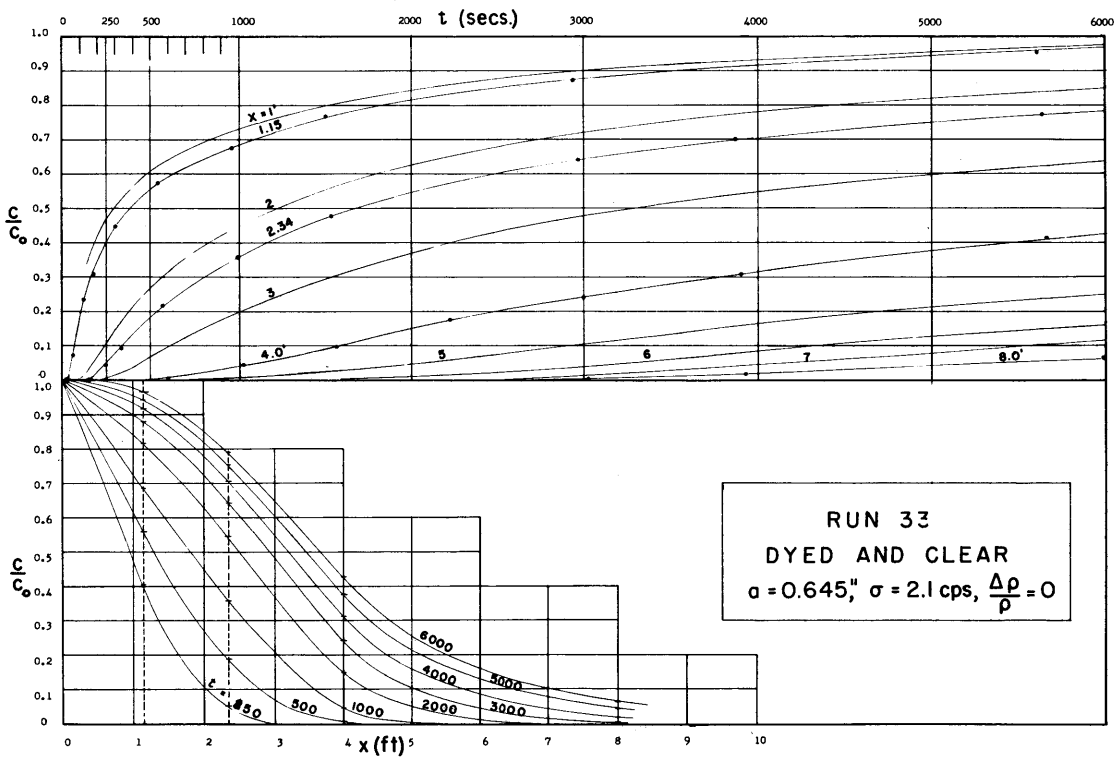
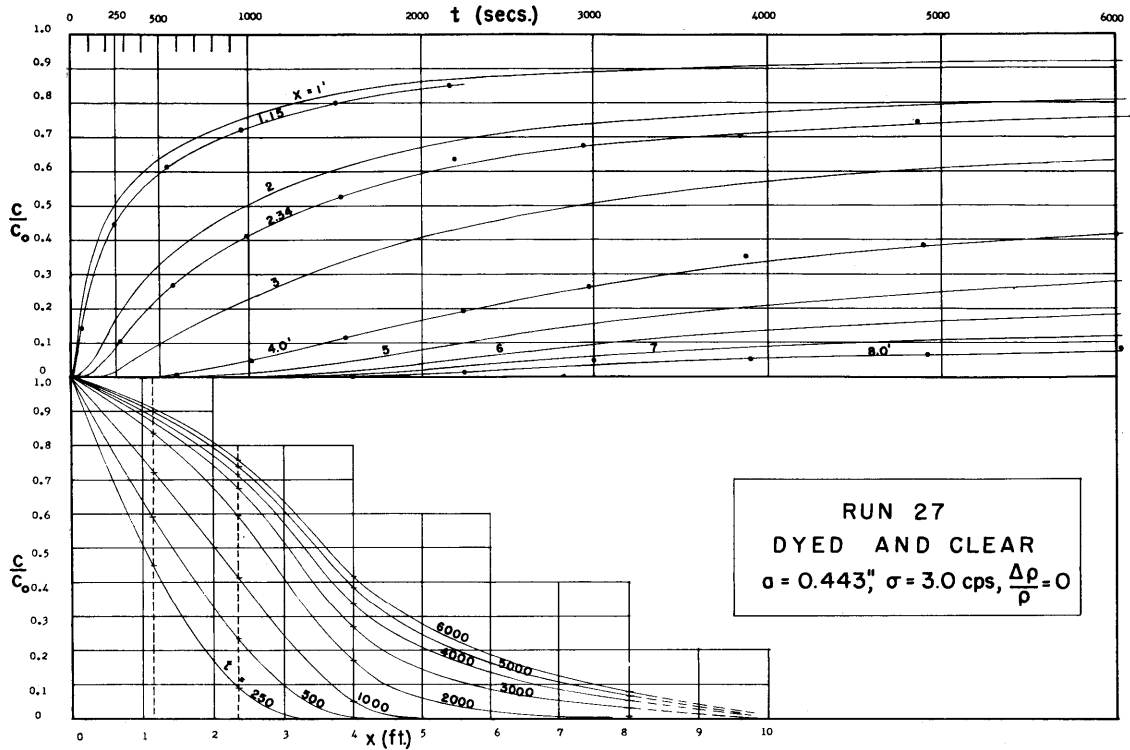
Plot 4. Mean rate of energy dissipation (input horsepower) as a function of turbulence parameter $a\sigma$ (amplitude times frequency of forced oscillation). Derived equation (straight line): $HP = 136 (a\sigma)^3$.



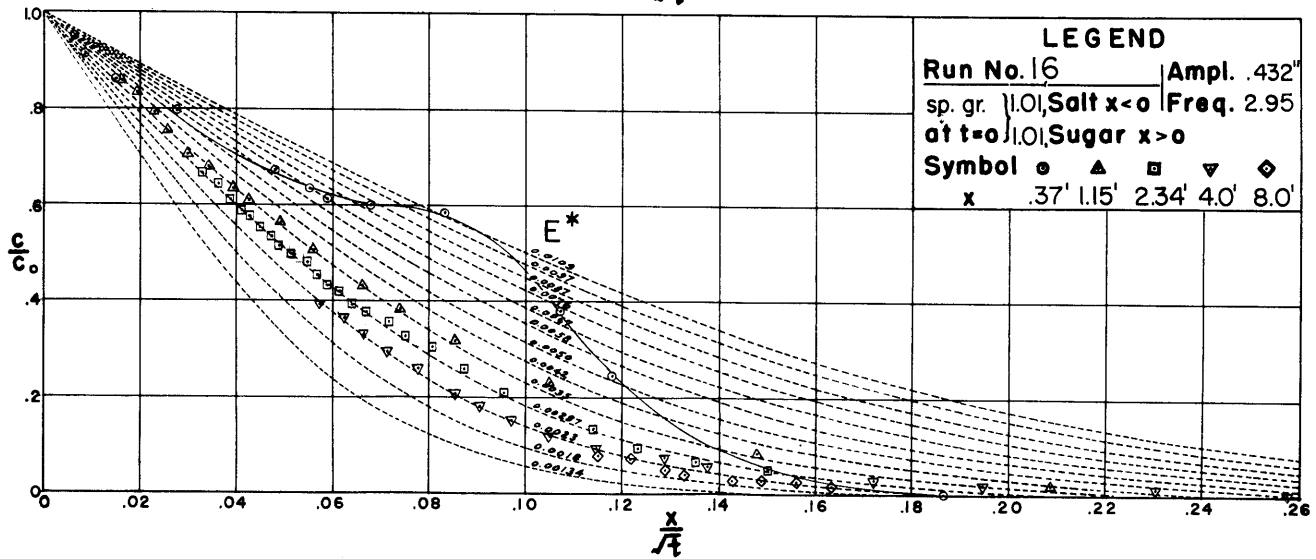
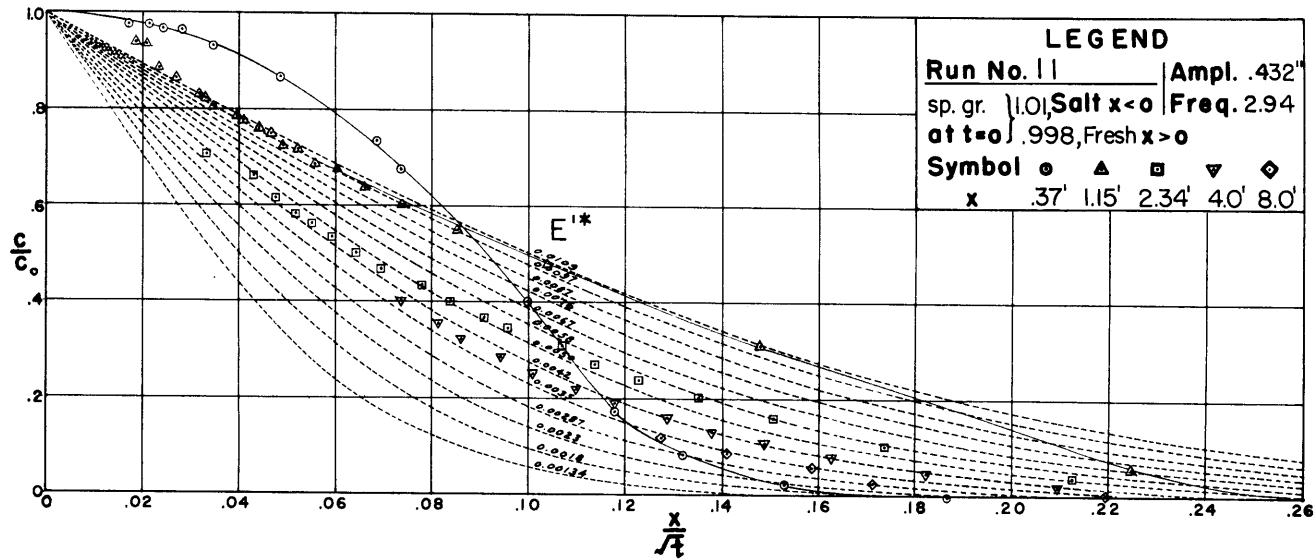
Plot 5. Concentration versus time and distance for typical time-dependent diffusion tests. Salt water diffusing into fresh water, differential specific gravity 1%.



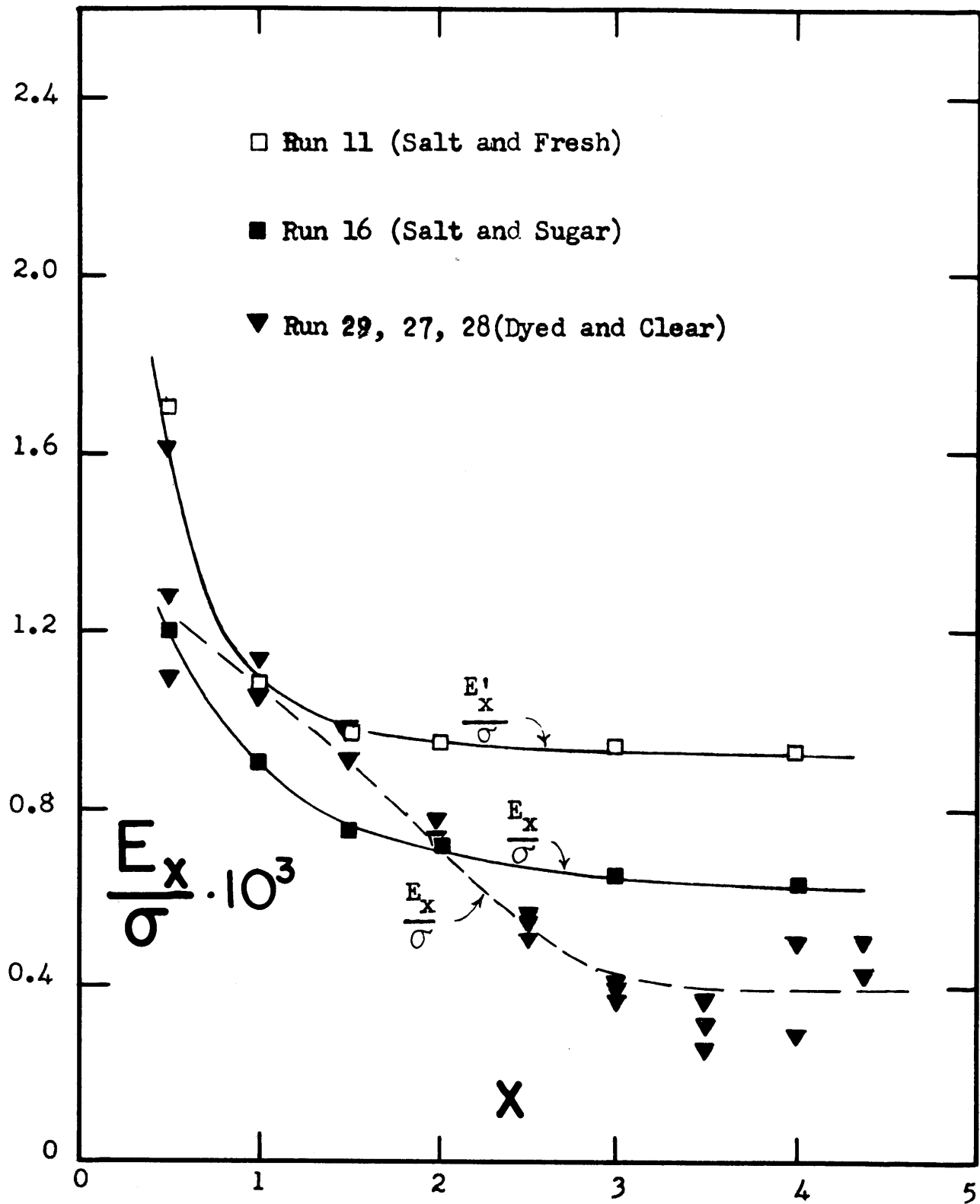
Plot 6. Concentration versus time and distance for typical time-dependent diffusion tests. Salt water diffusing into sugar water of equal specific gravity.



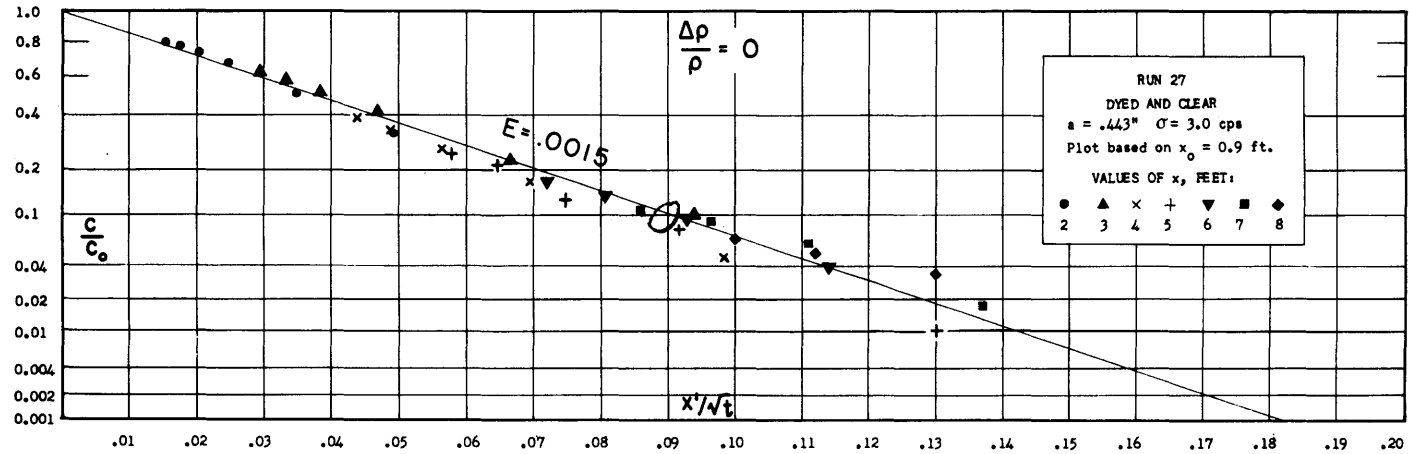
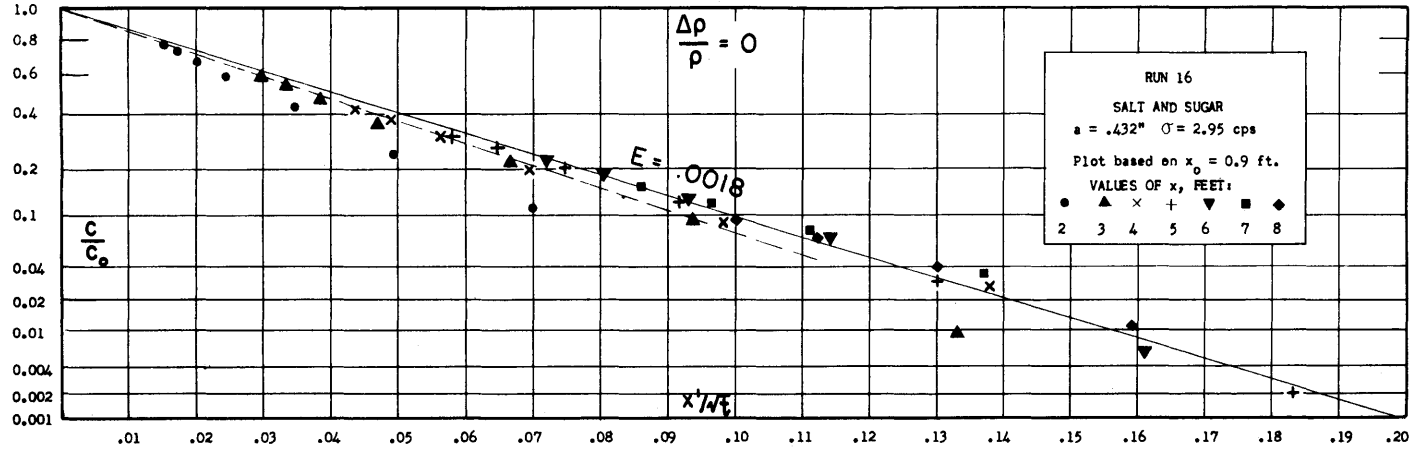
Plot 7. Concentration versus time and distance for typical time-dependent diffusion tests. Dyed fresh water diffusing into clear fresh water.



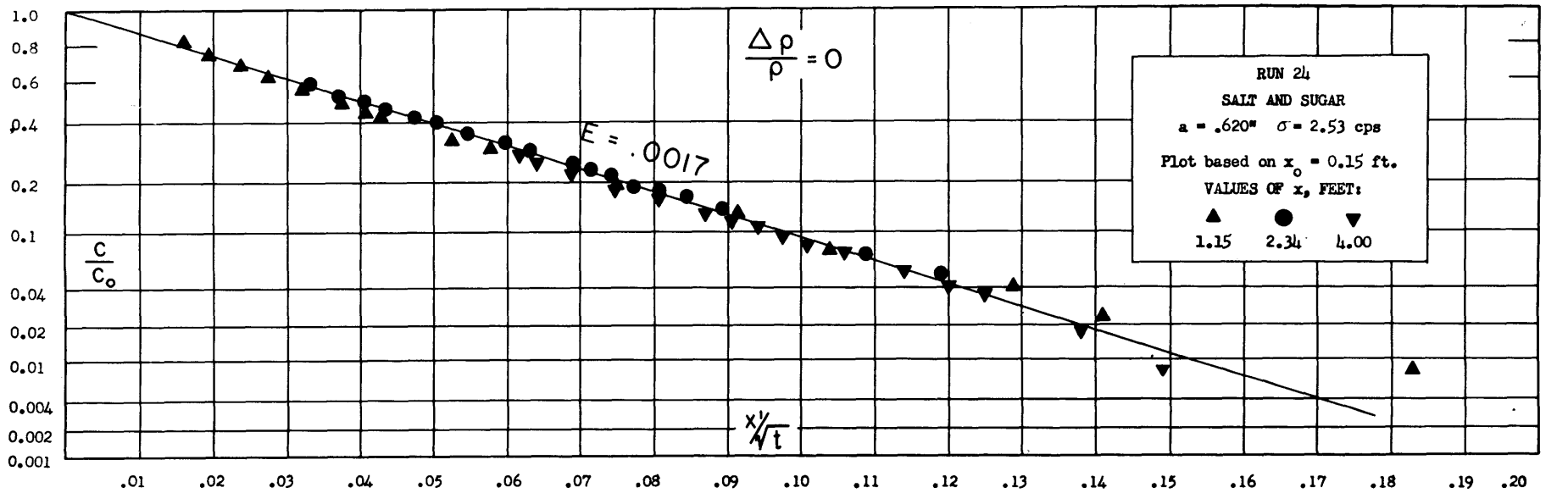
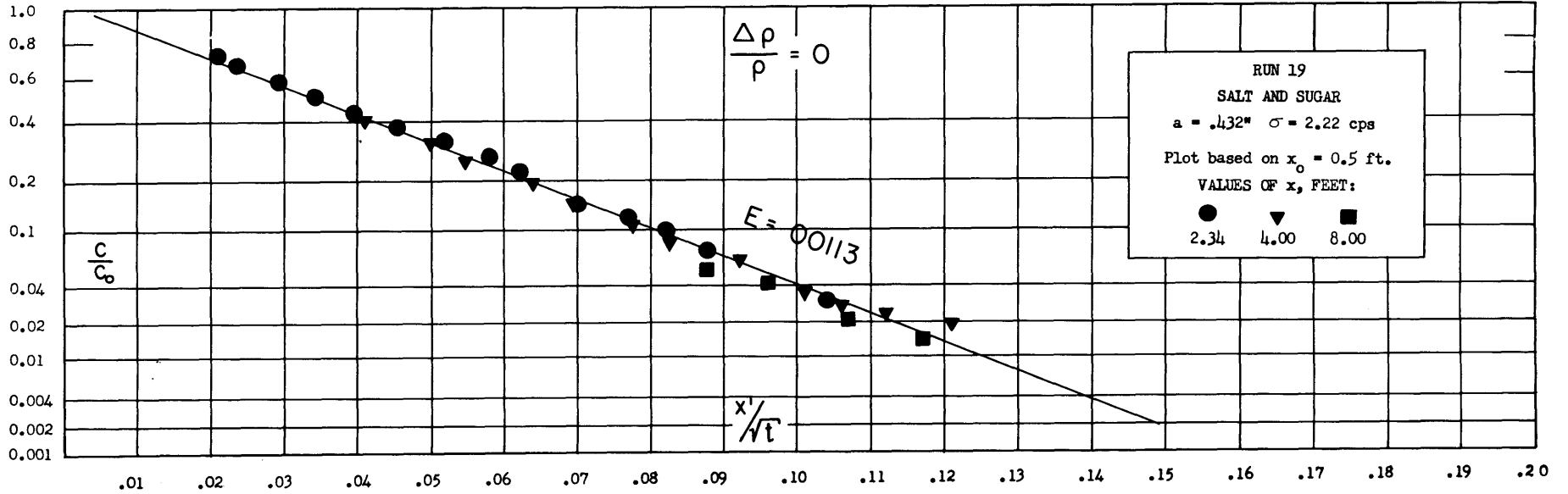
Plot 8. Concentration versus parameter x / \sqrt{t} , compared for typical time-dependent diffusion tests with and without initial density differential ($a = .432, \sigma = 2.9$, for both runs). Determination of E'^* and E^* .



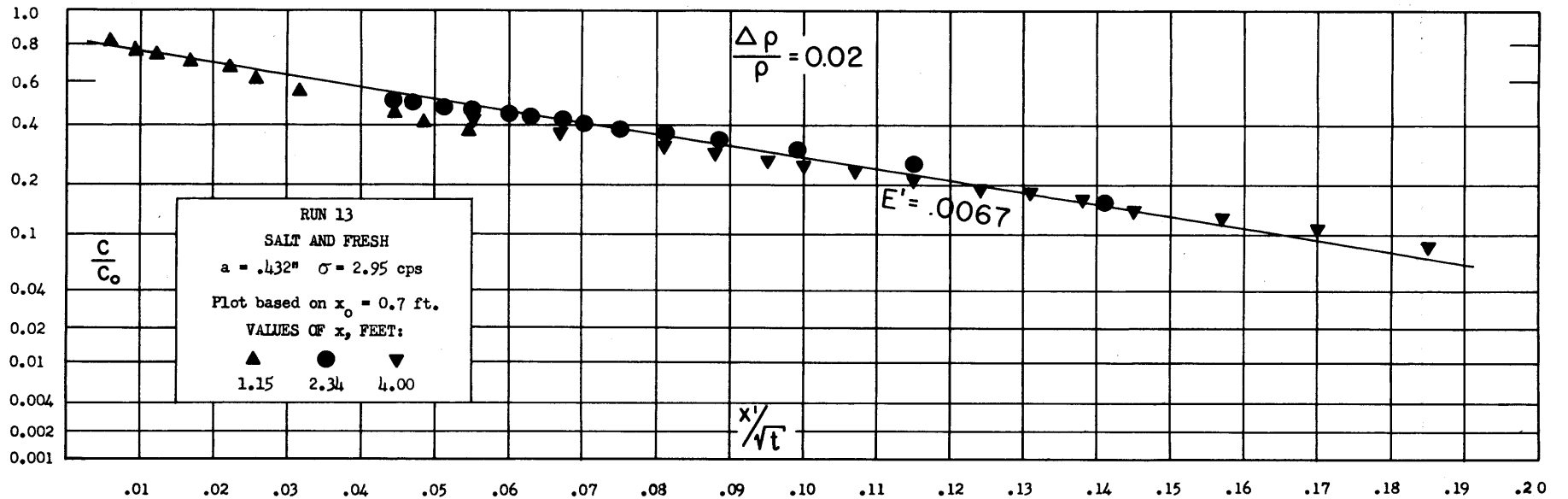
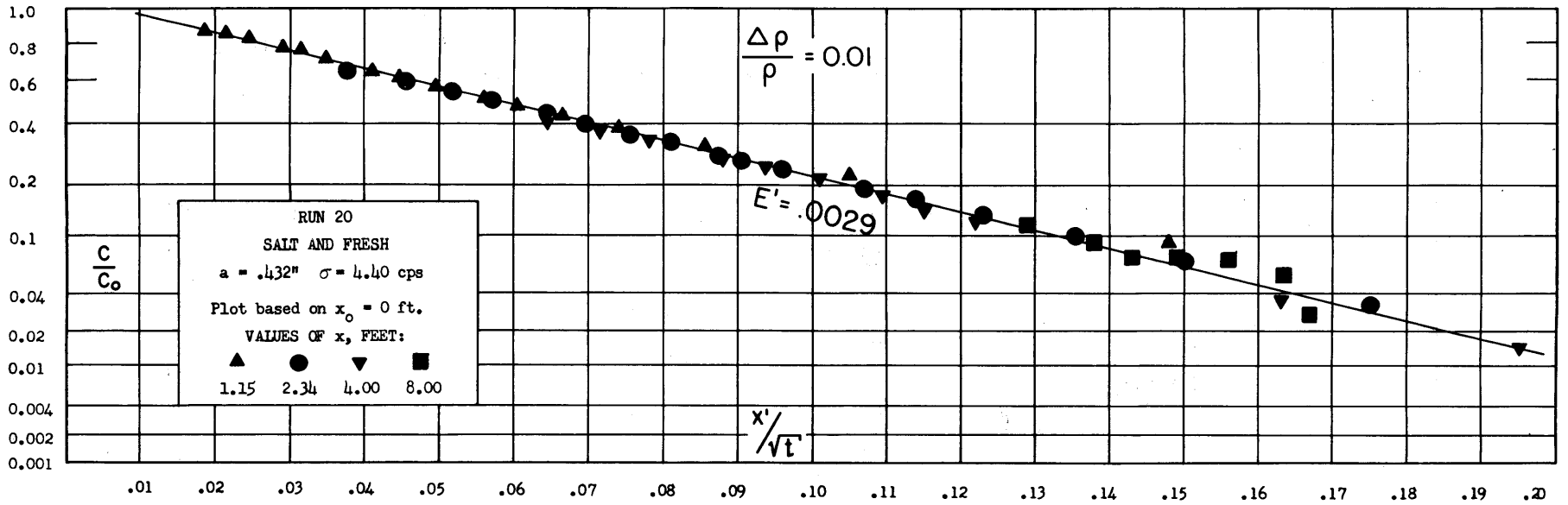
Plot 9. Local diffusion coefficient (E'_x or E_x) as a function of distance, x , from origin for typical time dependent diffusion tests.



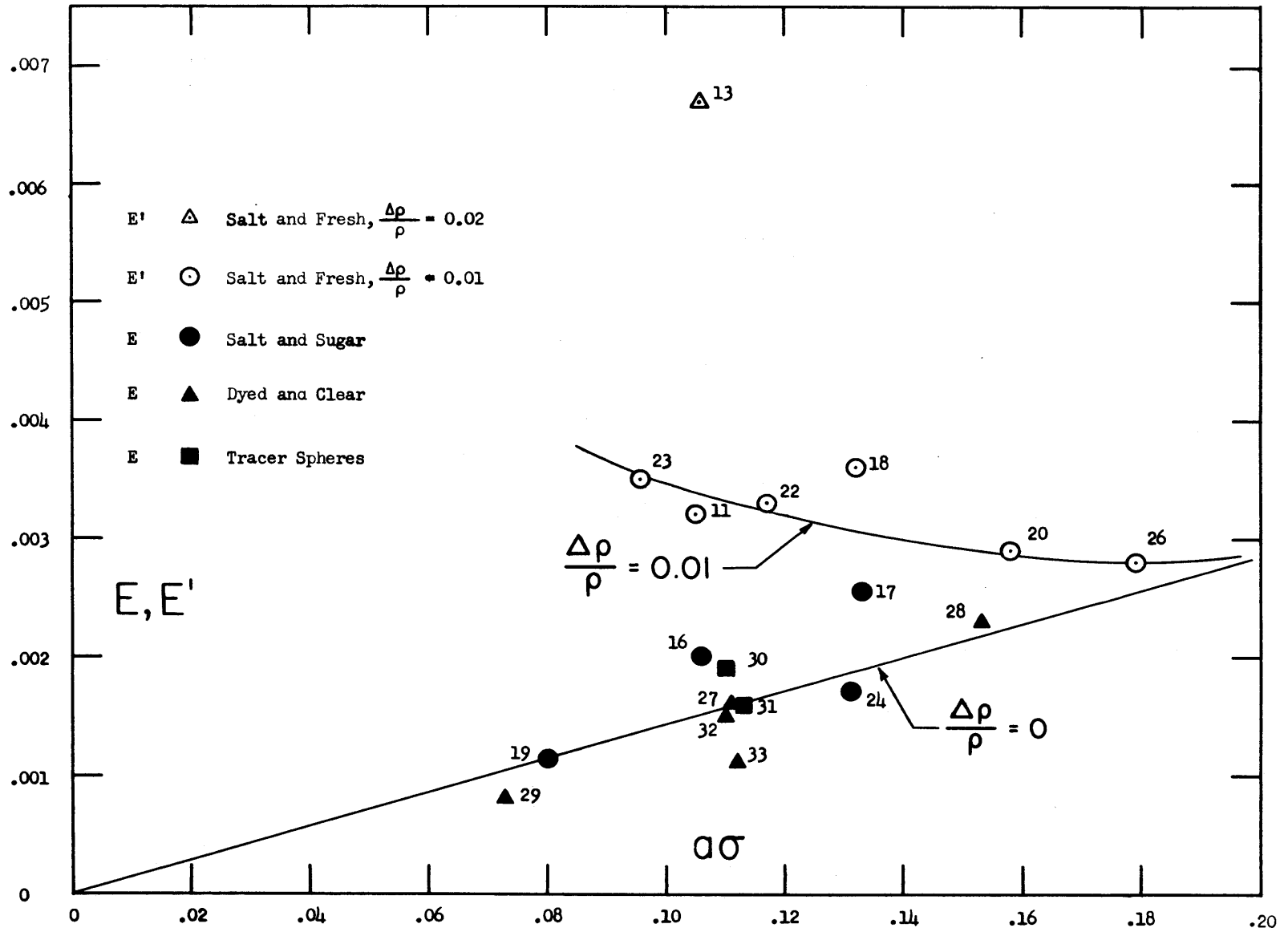
Plot 10. Concentration on probability scale versus parameter x'/\sqrt{t} for typical time-dependent diffusion tests without density differential. Comparison of (a) Salt and Sugar with (b) Dye ($a = .432$, $\sigma \approx 2.95$, for both runs).



Plot 11. Concentration on probability scale versus x'/\sqrt{t} for typical time dependent diffusion test without density difference.



Plot 14. Concentration on probability scale versus x'/\sqrt{t} for typical time dependent diffusion test with density difference.



Plot 15. Diffusion coefficients, E and E' , versus turbulence parameter, $a\sigma$, for time dependent diffusion test.

PUBLICATIONS BY THE STAFF OF THE
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