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Citation: Deshpande, A., S.E. Sarma, and V.K. Goyal. "Generalized Regular Sampling of Trigonometric Polynomials and Optimal Sensor Arrangement." IEEE Signal Processing Letters 17.4 (2010): 379–382. Web. 5 Apr. 2012. © 2010 Institute of Electrical and Electronics Engineers

As Published: http://dx.doi.org/10.1109/lsp.2010.2041962

Publisher: Institute of Electrical and Electronics Engineers (IEEE)

Persistent URL: http://hdl.handle.net/1721.1/69954

Version: Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

Generalized Regular Sampling of Trigonometric Polynomials and Optimal Sensor Arrangement

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***Abstract*—We address the *optimal sensor arrangement problem*, which is the determination of a geometric configuration of sensors such that the mean-squared error (MSE) in the estimation of an unknown trigonometric polynomial is minimum. Unsurprisingly, an arrangement in which sensors are spaced uniformly in each dimension is optimal. However, for multidimensional problems the minimum MSE is achieved with a much larger class of configurations that we call *generalized regular arrangements*. These arrangements are not necessarily generated by lattices and may exhibit great nonuniformity locally.**

***Index Terms*—Bandlimited signals, harmonic frames, multidimensional sampling, nonuniform sampling, sensor networks, tight frames.**

I. Introduction

Using sensing modalities including temperature, pressure, vibrations and chemical concentration levels, wireless sensor networks can provide measurements through which spatially-varying quantities can be estimated throughout a region of interest. In this letter, we consider the problem of estimating a bandlimited field from noisy local sample values of a physical quantity. Our interest is in how the *arrangement* of sensors affects the mean-squared error (MSE) of the field estimate, and we focus on finding a set of arrangements that are optimal under certain conditions on the noise and estimation procedure.

Absent noise and information other than an upper bound on the bandwidth in each dimension, the reconstruction problem has an exact solution under conditions analogous to the Nyquist condition for sampling of 1-D signals [1]. However, placing sensors precisely on a separable grid may not be practical. Spatial nonuniformity need not increase the number of samples measured, but it will generally make the reconstruction problem more difficult and more sensitive to noise; see [2]–[4] and references therein.

Our contribution is to define a large class of *generalized regular arrangements* that achieve the minimum MSE. These arrangements include ones that are not generated by lattices and may seem surprisingly uneven. We employ the formalism of frames [5], [6] and show that optimality of a sensor arrangement is equivalent to the tightness of an associated frame. We then show that certain transformations do not affect frame tightness (and hence arrangement optimality). The results seem to be novel in both the frame and sampling literature.

The most closely-related prior work is on the equivalent problem of "learning" trigonometric polynomials. Sugiyama and Ogawa [7] show that having uniformly-spaced samples in each spatial dimension is optimal. They also show that rigid translations of this regular arrangement or superpositions of two or more translated regular arrangements are optimal under certain conditions on the number of samples. These results are special cases of our more general construction. Moreover, once the connection to frame theory is made, they follow from well-known constructions of tight frames.

We formalize our problem in Section II, then review relevant results from frame theory in Section III. The solution of our sensor arrangement problem, presented in Section IV, comes from constructing a novel generalization of harmonic tight frames for multiple dimensions.

II. Problem Formulation

Let $f(\mathbf{x}) : \mathcal{T}^d \rightarrow \mathbb{R}$ denote the unknown scalar field to be estimated where $\mathcal{T}^d$ indicates a d-dimensional *toroidal* domain of unit length $[0,1]^d$. We assume that $f(\mathbf{x})$ is a trigonometric polynomial. This model is precisely equivalent to using the domain $[0,1]^d$ and applying periodic boundary conditions. Also, any bounded domain can be scaled and smoothly windowed to be approximated arbitrarily well by this model without any periodicity assumption. Though our developments hold for any dimension d, we limit most expressions and all examples to the case of $d = 2$.

The bandlimited assumption on the scalar field implies that f has the form

$$f(x,y) = \sum_{k=-M_1}^{+M_1} \sum_{\ell=-M_2}^{+M_2} a(k,\ell) \frac{e^{j2\pi(kx+\ell y)}}{\sqrt{(2M_1+1)(2M_2+1)}} \quad (1)$$

when $d = 2$, revealing $(2M_1+1)(2M_2+1)$ unknown coefficients and a set of $(2M_1+1)(2M_2+1)$ orthonormal basis functions. For a general treatment, let $D = \prod_{k=1}^{d}(2M_k+1)$, let $\phi_i(\mathbf{x})$ denote the ith basis function, and let a_i denote the ith unknown coefficient. Then (1) is expressed more abstractly as $f(\mathbf{x}) = \sum_{i=1}^{D} a_i \phi_i(\mathbf{x})$.

Denote the location of sensor n by $\mathbf{x}_n$. The measurement of sensor n is corrupted by additive noise ε_n, yielding the measurement model

$$g_n = f(\mathbf{x}_n) + \varepsilon_n, \qquad n = 1, 2, \ldots, N.$$

Manuscript received November 23, 2009; revised January 06, 2010. First published February 02, 2010; current version published February 19, 2010. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Yuriy V. Zakharov.

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Digital Object Identifier 10.1109/LSP.2010.2041962

We assume that $\{\varepsilon_n\}_{n=1}^N$ is a set of zero-mean, uncorrelated random variables with common variance σ^2.

Using the vector and matrix notations

$$\mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_D \end{bmatrix}, \quad \text{and}$$

$$\mathbf{V} = \begin{bmatrix} \phi_1(\mathbf{x}_1) & \phi_2(\mathbf{x}_1) & \cdots & \phi_D(\mathbf{x}_1) \\ \phi_1(\mathbf{x}_2) & \phi_2(\mathbf{x}_2) & \cdots & \phi_D(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{x}_N) & \phi_2(\mathbf{x}_N) & \cdots & \phi_D(\mathbf{x}_N) \end{bmatrix}$$

the field and measurement models yield

$$\mathbf{g} = \mathbf{V}\mathbf{a} + \mathbf{e}. \tag{2}$$

Matrix $\mathbf{V}$ is referred to as the *observation matrix*, and it depends on the *sensor arrangement* $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$.

Our assumptions create a non-Bayesian parameter estimation problem that is solved by the *minimum variance unbiased estimator* (MVUE) [8]

$$\hat{\mathbf{a}} = (\mathbf{V}^*\mathbf{V})^{-1}\mathbf{V}^*\mathbf{g}. \tag{3}$$

This estimator has MSE given by

$$\mathrm{MSE} = (1/D)\mathrm{E}\left[\|\hat{\mathbf{a}} - \mathbf{a}\|^2\right] = (\sigma^2/D)\mathrm{trace}(\mathbf{V}^*\mathbf{V})^{-1} \tag{4}$$

where $\mathbf{V}^*$ denotes the Hermitian transpose of $\mathbf{V}$. As the MSE is a function of the sensor arrangement X, we denote it as $\mathrm{MSE}(X)$. The optimal sensor arrangement problem is to find solutions to

$$X_{\mathrm{opt}} = \arg\min_{|X|=N} \mathrm{MSE}(X)$$

where $|X|$ denotes the number of sensor locations in X. Sugiyama and Ogawa [7] refer to the same problem as the optimal sample design for learning trigonometric polynomials.

III. Frame Review

A set of N vectors $\{\boldsymbol{\varphi}_n\}_{n=1}^N \subset \mathbb{C}^D$ is called a *frame* if

$$A\|\mathbf{a}\|^2 \leq \sum_{n=1}^{N} |\langle \mathbf{a}, \varphi_n \rangle|^2 \leq B\|\mathbf{a}\|^2, \quad \text{for all } \mathbf{a} \in \mathbb{C}^D \tag{5}$$

for some constants $A > 0$ and $B < \infty$ called the *frame bounds*. With a *tight frame* (TF), one can choose $A = B$. If $\|\boldsymbol{\varphi}_k\|^2 = 1$ for every k, then it is called a *unit-norm frame* (UNF). Three elementary facts about TFs that we will use are

1) if $U \in \mathbb{C}^{D \times D}$ is a unitary matrix, then $\{\varphi_n\} \subset \mathbb{C}^D$ is a TF if and only if $\{U\varphi_n\}$ is a TF;
2) the union of two TFs is also a TF; and
3) the tensor product of two TFs, similar to the tensor product of vector spaces, gives a TF.

The *analysis frame operator* $\mathbf{F}$ is an $N \times D$ matrix whose rows are conjugate transposes of the vectors $\boldsymbol{\varphi}_n$. It maps a vector $\mathbf{a} \in \mathbb{C}^D$ into a vector of *frame coefficients* $\mathbf{g} \in \mathbb{C}^N$:

$$\mathbf{g}_n = (\mathbf{F}\mathbf{a})_n = \langle \mathbf{a}, \varphi_n \rangle, \quad \text{for } n = 1, 2, \ldots, N.$$

$\mathbf{F}^*\mathbf{F}$ is referred to as the *frame operator*, and the lower bound of (5) ensures that it is invertible. For a TF, $\mathbf{F}^*\mathbf{F} = A\mathbf{I}_D$.

Our use of frame theory is transparent from the reuse of the notations $\mathbf{a}$ and $\mathbf{g}$: we see the observation matrix as an analysis frame operator. The following theorem describes both the optimal estimates (consistent with the development in Section II) and which observation matrices are optimal.

Theorem 1 ([9]): Consider the estimation of $\mathbf{a}$ from noisy frame coefficients

$$\hat{\mathbf{y}} = \mathbf{F}\mathbf{a} + \mathbf{e} \tag{6}$$

where $\mathbb{E}[\mathbf{e}] = 0$ and $\mathbb{E}[\mathbf{e}^T\mathbf{e}] = \sigma^2\mathbf{I}_N$. The estimate

$$\hat{\mathbf{a}} = \mathbf{F}^\dagger\hat{\mathbf{y}} \tag{7}$$

minimizes the MSE defined as $(1/D)\mathbb{E}\|\hat{\mathbf{a}} - \mathbf{a}\|^2$, where $\mathbf{F}^\dagger$ denotes the pseudoinverse of $\mathbf{F}$ and is given by $\mathbf{F}^\dagger = (\mathbf{F}^*\mathbf{F})^{-1}\mathbf{F}^*$. For any frame, the MSE satisfies

$$\sigma^2/B \leq \mathrm{MSE} \leq \sigma^2 A. \tag{8}$$

For a unit-norm frame (UNF),

$$\sigma^2 D/N \leq \mathrm{MSE} \leq \sigma^2 A. \tag{9}$$

A UNF is tight if and only if

$$\mathrm{MSE} = \sigma^2 D/N. \tag{10}$$

IV. Regular Sampling and Optimal MSE

Consider a frame formed by N vectors of the form $\mathbf{v}_n = [\phi_1^*(\mathbf{x}_n)\phi_2^*(\mathbf{x}_n)\ldots\phi_D^*(\mathbf{x}_n)]^{\mathrm{T}}$. The observation matrix $\mathbf{V}$ is the corresponding analysis frame operator. The frame is a UNF since $\|\mathbf{v}_n\|^2 = 1$ for each n. As a consequence of this and Theorem 1, we get the following result.

Corollary 2: X_{opt} is an optimal sensor arrangement if and only if it leads to a TF, in which case

$$\mathrm{MSE}(X_{\mathrm{opt}}) = \frac{\sigma^2 D}{N}. \tag{11}$$

While there are several mechanisms for findings sets of tight frames [9]–[11], the difficulty of our problem arises from the constraint that the frame vectors have forms fixed by the sampling of a trigonometric polynomial (1) (or its equivalent for higher dimensions). No full characterization of such tight frames is known; we provide novel sufficient conditions.

In 1-D, regular (uniform) sensor arrangement leads to a TF and hence to minimum MSE. Specifically, placing the nth sensor at location $(n-1)/N$ for $n = 1, 2, \ldots, N$ and using the 1-D analogue of field model (1), the analysis frame operator is given by

$$\mathbf{F}_{n,k} = \frac{1}{\sqrt{2M+1}} e^{j2\pi k(n-1)/N}$$

for $n = 1, 2, \ldots, N$ and $k = -M, -M+1, \ldots, M$. The associated frame is both a TF and a UNF. Moreover, it is an example of a *harmonic frame* [9].

Shifting all sensors by the same amount (modulo the toroidal boundary condition) is equivalent to multiplying $\mathbf{F}$ by a unitary matrix; hence it does not affect tightness or optimality of MSE. Since the tensor product of TFs is a TF, it follows that regular

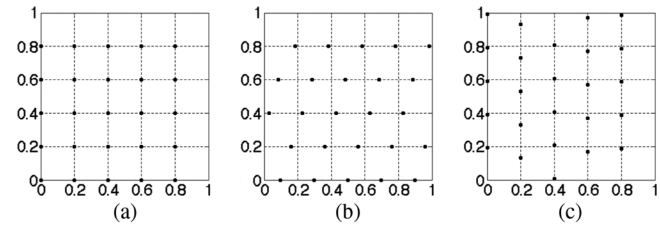

Fig. 1. (a) Regular arrangement of 5×5 sensors. (b) Independent line translations along x-axis. (c) Independent line translations along y-axis.

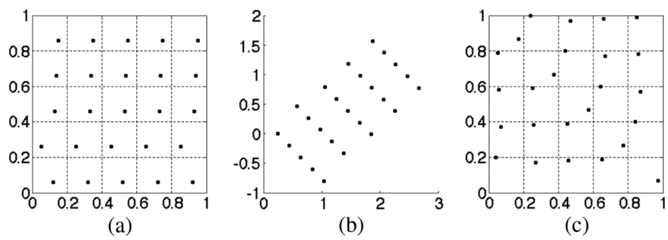

Fig. 2. (a) Independent line translations along x-axis and a rigid translation. (b) Integer linear transform with $a = 1, b = 2, c = -1$ and $d = 2$ for $M = 1$. (c) Mapping all the sensor locations back to $[0, 1]^2$.

arrangements in higher dimensions obtained by regular arrangements in 1-D are optimal sensor arrangements. Furthermore, the union of regular arrangements yields a union of TFs and again optimality is maintained. These optimality results that follow from frame theory are equivalent to results of Sugiyama and Ogawa [7]. Sugiyama and Ogawa also show that arrangements obtained by rigid translation of a regular arrangement (modulo the toroidal boundary conditions) are optimal.

We provide in the following subsection more general variations on regular arrangements that also yield the minimum MSE. Note in particular that these fall outside all equivalences defined in [9], [11]. We refer to them as the *generalized regular arrangements*.

A. Generalized Regular Arrangements in 2-D

Let us start with a regular arrangement of $N = N_1 N_2$ sensors, where each $N_i \geq (2M_i + 1)$ and the coordinates of the sensors are $((i_1 - 1)/N_1, (i_2 - 1)/N_2)$ with $i_1 = 1, 2, \ldots, N_1$ and $i_2 = 1, 2, \ldots, N_2$. We propose two geometric transformations to obtain generalized regular arrangements.

Transformation 1) Independent Line Translations Along an Axis: We perform this transformation with respect to *one* chosen axis. If we choose x-axis (alternatively y-axis), we independently translate each group of N_1 (or N_2) sensor locations with the same y (or x) coordinate by some distance along the x-axis (or y-axis). We map the sensor locations that fall out of $[0, 1]^2$ back to the domain using the periodic boundary conditions. We call this transformation *independent line translations* along the x-axis (or y-axis). In Fig. 1, we show a 5×5 regular arrangement of sensors and illustrate independent line translations along either axis.

Transformation 2) Integer Linear Transform: We can perform this transformation after carrying out Transformation 1 and a 2-D rigid translation of the entire sampling set. We present the case involving independent line translations along the x-axis. The integer linear transform is a special type of linear transform in which we map each sensor location (x, y) to $(ax + by, cx + dy)$, where a, b, c and d are all integers such that the following three conditions are satisfied.

i) $ka \neq 0 (\mathrm{mod}\ N_1)$ for every $k \in \{1, 2, \ldots, 2M_1\}$.
ii) $\ell c \neq 0 (\mathrm{mod}\ N_1)$ for every $\ell \in \{1, 2, \ldots, 2M_2\}$.
iii) For every

$$k \in \{-2M_1, -2M_1 + 1, \ldots, -1, 1, 2, \ldots, 2M_1 - 1, 2M_1\},$$
$$\ell \in \{-2M_2, -2M_2 + 1, \ldots, -1, 1, 2, \ldots, 2M_2 - 1, 2M_2\}$$

$ka + \ell c$ is not a nonzero integer multiple of N_1 and at least one of the following always holds.
- $ka + \ell c \neq 0$;
- $kb + \ell d \neq 0 (\mathrm{mod}\ N_2)$.

If we had performed independent line translations along the y-axis, the above conditions remain the same, except we need to switch between a and b, c and d, and N_1 and N_2. We map all the sensor locations that fall out of $[0, 1]^2$ back to the domain. This transformation is illustrated in Fig. 2.

We call any sensor arrangement obtained using the above transformations a *generalized regular arrangement* in 2-D.

Theorem 3: A generalized regular arrangement of $N = N_1 \times N_2$ sensors in 2-D, where $N_1 \geq (2M_1 + 1)$ and $N_2 \geq (2M_2 + 1)$, is an optimal sensor arrangement.

Proof: Consider a periodic regular 2-D arrangement of $N_1 \times N_2$ sensors. After carrying out Transformation 1 (along the x-axis), a 2-D rigid translation and Transformation 2, a sensor location in the resulting arrangement is of the form $(a\alpha_{i_1 i_2} + b\beta_{i_1 i_2}, c\alpha_{i_1 i_2} + d\beta_{i_1 i_2})$ where

$$\alpha_{i_1 i_2} = \left(\frac{i_1 - 1}{N_1} + \Delta_{i_2} + \Delta_x\right), \ \beta_{i_1 i_2} = \left(\frac{i_2 - 1}{N_2} + \Delta_y\right),$$
$$i_1 = 1, 2, \ldots, N_1 \text{ and } i_2 = 1, 2, \ldots, N_2$$

and (Δ_x, Δ_y) represents a 2-D rigid translation and Δ_{i_2}s represent distances in line translations along the x-axis.

For the 2-D case, $D = (2M_1 + 1)(2M_2 + 1)$. Let $\mathbf{T} = \mathbf{V}^* \mathbf{V}$. Each element of T is of the form

$$T_{k,\ell,m,n} = \frac{1}{D} \sum_{i_2=1}^{N_2} \sum_{i_1=1}^{N_1} e^{j2\pi\left[(k-\ell)x_{i_1 i_2} + (m-n)y_{i_1 i_2}\right]}$$

where $k, \ell \in \{-M_1, -M_1 + 1, \ldots, 0, \ldots, M_1 - 1, M_1\}$, and $m, n \in \{-M_2, -M_2 + 1, \ldots, 0, \ldots, M_2 - 1, M_2\}$, forming D^2 elements of $\mathbf{T}$. The sensor locations are $(x_{i_1 i_2}, y_{i_1 i_2})$ with $i_1 = 1, 2, \ldots, N_1$ and $i_2 = 1, 2, \ldots, N_2$. Their forms are given above. Substituting and rearranging, we get

$$T_{k,\ell,m,n} = \frac{1}{D} e^{j2\pi[(k-\ell)(a\Delta_x + b\Delta_y) + (m-n)(c\Delta_x + d\Delta_y)]}$$
$$\cdot \sum_{i_2=1}^{N_2} e^{j2\pi[(k-\ell)b + (m-n)d]\left(\frac{i_2 - 1}{N_2}\right)} e^{j2\pi[(k-\ell)a + (m-n)c]\Delta_{i_2}}$$
$$\cdot \sum_{i_1=1}^{N_1} e^{j2\pi[(k-\ell)a + (m-n)c]\left(\frac{i_1 - 1}{N_1}\right)}.$$

The first factor on the right hand side of the equality can be interpreted as a 2-D rigid translation of the entire sampling set by $(a\Delta_x + b\Delta_y, c\Delta_x + d\Delta_y)$. We have already shown that a rigid translation does not change $\mathbf{T}$. Thus we focus on the remaining factors. We deal with four different cases.

- *Case 1*: $k = \ell$ and $m = n$. Thus, $T_{k,\ell,m,n} = N_1 N_2 D$.

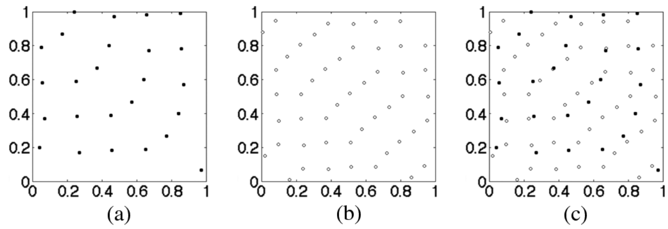

Fig. 3. (a) and (b) Arrangements of 5×5 and 7×7 sensors obtained after using Transformation 1, 2-D rigid translation, and Transformation 2. (c) Superposition of arrangements shown in (a) and (b).

- *Case 2*: $k = \ell$ and $m \neq n$. According to the second condition in Transformation 2, $(m-n)c \neq 0(\mathrm{mod} N_1)$. Hence, $\sum_{i_1=1}^{N_1} e^{j2\pi(m-n)c(i_1-1)/N_1} = 0$. Thus, $T_{k,\ell,m,n} = 0$.
- *Case 3*: $m = n$ and $k \neq \ell$. According to the first condition in Transformation 2, $(k-\ell)a \neq 0(\mathrm{mod}\ N_1)$. Hence, $\sum_{i_1=1}^{N_1} e^{j2\pi(k-\ell)a(i_1-1)/N_1} = 0$. Thus, $T_{k,\ell,m,n} = 0$.
- *Case 4*: $k \neq \ell$ and $m \neq n$. According to the third condition of Transformation 2, $(k-\ell)a + (m-n)c \neq iN_1$ for any nonzero integer i. If $(k-\ell)a + (m-n)c \neq 0$, then $\sum_{i_1=1}^{N_1} e^{j2\pi[(k-\ell)a+(m-n)c](i_1-1)/N_1} = 0$. Thus, $T_{k,\ell,m,n} = 0$. If $(k-\ell)a + (m-n)c = 0$, then according to the third condition of Transformation 2, $kb + \ell d \neq 0(\mathrm{mod}\ N_2)$. Therefore, $\sum_{i_2=1}^{N_2} e^{j2\pi[(k-\ell)b+(m-n)d](i_2-1)/N_2} = 0$. Again, $T_{k,\ell,m,n} = 0$.

Combining the cases, $\mathbf{T} = (N_1 N_2/D)\mathbf{I}$. Thus, the generalized regular arrangement obtained using the transformations above lead to a tight frame and hence yields the optimal MSE. ■

Besides the above transformations, the superposition of two generalized regular arrangements also yields the optimal MSE because the union of two TFs is a TF. Sugiyama and Ogawa [7] restrict superposition of translated regular arrangements to a special case where the starting regular arrangement has a number of samples equal to the Nyquist rate. Fig. 3 shows an optimal sensor arrangement obtained by superposing two generalized regular arrangements of 5×5 and 7×7 sensors. This shows that an optimal arrangement can be superficially uneven and can have sensors arbitrarily close together. Fig. 3 is suggestive of the conjecture that near any unit-norm frame there is a unit-norm tight frame. The related question of constructing a unit-norm tight frame near any given frame is addressed in [12].

B. Generalized Regular Arrangements in Higher Dimensions

We briefly comment on generalized regular arrangements in the d-dimensional domain $[0,1]^d$. Let $N = N_1 \times N_2 \times \cdots \times N_d$ be the number of sensors, where each $N_k \geq (2M_k+1)$. Let $x_1, x_2, \ldots, x_d$ denote the axes. Again, we start with a regular periodic arrangement in the d-dimensional domain. We modify Transformation 1 with *hierarchical hyperplane translations* along different axes. At the first level, we fix an axis; say x_d. There are N_d hyperplanes of dimension $d-1$ each containing regular arrangements of size $N_1 \times N_2 \times \cdots \times N_{(d-1)}$. We independently translate each of these arrangements along the x_d-axis with possibly different distances. At the second level, we fix another axis, say $x_{(d-1)}$, and deal with $N_d \times N_{(d-1)}$ hyperplanes of dimension $d-2$ containing regular arrangements. We translate these arrangements along the $x_{(d-1)}$-axis with possibly different distances. We continue in this way till we deal with every axis. Note that the order in which we choose axes is not important. We call this transformation hierarchical hyperplane translations. The second transformation involving integer linear transform involves multiplying sensor coordinates with a $d \times d$ matrix with special integer elements. We can obtain a set of conditions on the entries of this matrix similar to the 2-D case. We omit these details.

C. Open Question

Following the geometric transformations proposed earlier, we can show optimality of sensor arrangements with the number of samples N only of the form $N = \sum_{i=1}^{K} N_1^i N_2^i$, where K is a positive integer, $N_1^i \geq (2M_1+1)$ and $N_2^i \geq (2M_2+1)$. The question of finding optimal arrangements for N which cannot be expressed in the above form remains open. We can further reduce this question to finding optimal arrangements for the number of samples from the set $J = \{D+1, D+2, \ldots, 2-D-1\}$ except the terms which are not integer multiples of $(2M_1+1)$ or $(2M_2+1)$, where $D = (2M_1+1)(2M_2+1)$.

V. Conclusions

Regular sampling not only accommodates easy reconstruction of a band-limited signal from sample values but also minimizes the MSE of the field estimate under certain conditions on the noise and estimation procedure. We generalized regular sampling to a set of arrangements that are surprisingly uneven and yet possess these same properties as regular arrangements.

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